# THE COORDINATE RING OF A VARIETY

Reference: Sections 2.1, 2.2, 2.3 "Algebraic Curves", Fulton

From now on, $K$ will be a fixed algebraically closed field.

Def: An irreducible algebraic set of $\mathbb{A}^n(K)$ is called an affine variety.

## POLYNOMIAL MAPS

Let $V \subseteq \mathbb{A}^n(K)$ be a nonempty variety. We consider
$$\mathcal{F}(V, K) = \{ f : V \to K \}$$

the set of all functions from $V$ to $K$. Note that a function in $\mathcal{F}(V,K)$ may not be defined at all points of $V$. If $f$ is defined at a point $P \in V$, then $f(P) \in K$.

The set $\mathcal{F}(V,K)$ is a ring with the addition and multiplication defined in the following way:

for all $f, g \in \mathcal{F}(V,K)$ :
$$(f+g)(x) := f(x) + g(x),$$
$$(f \cdot g)(x) := f(x) \cdot g(x).$$

Note that $K \subseteq \mathcal{F}(V,K)$ is a subring which consists of all constant functions on $V$.

Def: A function $f \in \mathcal{F}(V,K)$ is called a polynomial function on $V$ if there is a polynomial $F \in K[x_1, \dots, x_n]$ such that.
$$f(a_1, \dots, a_n) = F(a_1, \dots, a_n) \quad \forall \ (a_1, \dots, a_n) \in V.$$

Note that a polynomial function $f$ is defined at all points $P$ of $V$. If $P(a_1, \dots, a_n)$ then we denote
$$f(P) := f(a_1, \dots, a_n) \in K.$$

e.g: Consider $V = V(y) \subseteq \mathbb{A}^2(K)$ and the function
$$f(x,y) = \frac{x}{y+1}.$$

Let $P(x,0) \in V$. We have:

$$f(P) = \frac{x}{0+1} = x,$$

So if $F(x,y) = x \in K[x,y]$, then $f(P) = F(P)$ $\forall P \in V$, which implies that $f$ is a polynomial function.

It is easy to show that the set of polynomial functions is a subring of $\mathcal{F}(V,K)$.

Every polynomial function can be represented by a polynomial $F \in K[X_1, \dots, X_n]$.

Now, it can happen that two different polynomials $F$ and $G$ of $K[X_1, \dots, X_n]$ define the same polynomial function, i.e:

$$F(P) = G(P) \quad \forall P \in V$$
$$\Updownarrow$$
$$F(P) - G(P) = 0 \quad \forall P \in V$$
$$\Updownarrow$$
$$(F-G)(P) = 0 \quad \forall P \in V$$
$$\Updownarrow$$
$$F - G \in I(V)$$
$$\Updownarrow$$
$$\overline{F} = \overline{G} \quad \text{in} \quad \frac{K[X_1, \dots, X_n]}{I(V)}$$

We notice that, since $V$ is a variety, then $I(V)$ is a prime ideal and $\frac{K[X_1, \dots, X_n]}{I(V)}$ is an integral domain.
This leads to the following definition.

## THE COORDINATE RING OF V

**Def**: Let $V$ be a nonempty variety. The coordinate ring of $V$, denoted by $K[V]$, is the finitely generated $K$-algebra defined as:

$$K[V] \cong \frac{K[X_1, \dots, X_n]}{I(V)}.$$

We have:

$$\left\{ \begin{array}{c} \text{polynomial} \\ \text{functions} \\ \text{on } V \end{array} \right\} \cong K[V]$$

e.g.: • If $V = \mathbb{A}^n(K) \Rightarrow I(V) = (0) \Rightarrow K[\mathbb{A}^n] = \frac{K[x_1, \dots, x_n]}{(0)} \cong K[x_1, \dots, x_n]$.

• If $V = V(y) \subseteq \mathbb{A}^2(K) \Rightarrow I(V) = (y) \Rightarrow K[V] = \frac{K[x,y]}{(y)} \cong K[x]$.

Remark: If $V$ is an algebraic set, which is not a variety, i.e. $V$ is reducible, then $K[V]$ has nonzero zero divisors:

e.g.: $V = V(xy) \Rightarrow K[V] = \frac{K[x,y]}{(xy)}$.

If $\bar{x} = x + (xy)$, $\bar{y} = y + (xy)$ then $\bar{x}, \bar{y} \neq \bar{0}$ but $\overline{xy} = \bar{0}$.

## THE EVALUATION HOMOMORPHISM

Let $P \in V$. For all $\bar{F} \in K[V]$ the value $\bar{F}(P) := F(P)$ is well defined since it does not depend on the representitive chosen for $\bar{F}$.

So we can define a homomorphism, called evaluation homomorphism in the following way:

$$\begin{aligned} \Theta_P : K[V] &\longrightarrow K \\ \bar{F} &\longmapsto F(P) \end{aligned}$$

$\Theta_P$ is clearly surjective, since $K \subseteq K[V]$. Moreover we have:

$$\mathrm{Ker}(\Theta_P) = \{ \bar{F} \in K[V] : F(P) = 0 \}.$$

Then, by the first isomorphism theorem we get:

$$\frac{K[V]}{\mathrm{Ker}(\Theta_P)} \cong K.$$

So $\mathrm{Ker}(\Theta_P)$ is a maximal ideal of $K[V]$ that we will denote $M_P(V)$

$$M_P(V) := \mathrm{Ker}(\Theta_P) = \{ \bar{F} \in K[V] : F(P) = 0 \}$$

## Example

Consider $V = V(y - x^2)$.

Let $F(x,y) = x - y \in K[x,y]$. We have $\overline{F} = \overline{x - y + (y - x^2)} \in K[V]$.

This also means that for all $H(x,y) \in (y - x^2)$ we have:

$$\overline{F+H} = \overline{F}.$$

So, for instance, if $H(x,y) = y - x^2$ the polynomial

$$G(x,y) = F(x,y) + H(x,y) = x - x^2$$

define the same polynomial function as $F(x,y)$, i.e. $\overline{G} = \overline{F}$.

Moreover for all $H(x,y) \in (y - x^2)$, for all $P \in V$ we have:

$$(F+H)(P) = F(P) + H(P) = F(P) + 0 = F(P) \quad .$$

Note that $K[V] \cong K[x]$. Indeed

$$\alpha : K[V] \longrightarrow K[x] \qquad , \qquad \alpha^{-1} : K[x] \longrightarrow K[V]$$

$$\overline{F(x,y)} \longmapsto F(x, x^2) = \tilde{F}(x) \qquad\qquad F(x) \longmapsto \overline{F(x)}$$

are isomorphisms between $K[V]$ and $K[x]$ inverse to each other.

Now, let $P(1,1) \in V$. We can consider the evaluation homomorphism

$$\begin{aligned} \theta_P : K[V] &\longrightarrow K \\ \overline{F(x,y)} &\longmapsto F(1,1) \end{aligned}$$

If $M_P(V) = \text{Ker}(\theta_P) = \{ \overline{F} \in K[V] : F(1,1) \}$, we have $M_P(V) = (\overline{x-1})$. Indeed, let $\overline{F} \in M_P$. Then $\alpha(\overline{F}) = \tilde{F}(x) = F(x, x^2)$ satisfies $\tilde{F}(1) = 0$, i.e. $(x-1) \mid \tilde{F}(x)$. This implies

$$\tilde{F}(x) = (x-1)\, G(x) \implies \overline{F} = \overline{(x-1)} \cdot \overline{G} \implies \overline{F} \in (\overline{x-1}).$$

Consequentely

$$\frac{K[V]}{(\overline{x-1})} \cong K.$$

# MORPHISMS BETWEEN VARIETIES

Let $V \subseteq \mathbb{A}^n(K)$ be a non empty variety.

Let $F_1, \dots, F_m \in K[V]$. We can consider the map:

$$\begin{aligned} \varphi : V &\longrightarrow \mathbb{A}^m(K) \\ P &\longmapsto (F_1(P), \dots, F_m(P)). \end{aligned}$$

We write $\varphi = (F_1, \dots, F_m)$.

Since for each $i$ the value $F_i(P)$ is well-defined, the map $\varphi$ is well-defined on $V$.

The map $\varphi$ is an example of "polynomial map" (or morphism) between $V$ and $\mathbb{A}^m(K)$.

More in general, we can consider similar maps between two varieties. We have the following definition:

<u>Def</u>: Let $V \subseteq \mathbb{A}^n(K)$, $W \subseteq \mathbb{A}^m(K)$ be non empty varieties.
A polynomial map or a morphism

$$\varphi : V \longrightarrow W$$

is a tuple $(F_1, \dots, F_m)$, with $F_i \in K[V]$ such that

$$\varphi(P) = (F_1(P), \dots, F_m(P)) \in W$$

for all $P \in V$.

<u>Remark</u>: In other terms, $\varphi : V \to W$ is a morphism if there exist polynomials $F_1, \dots, F_m \in K[X_1, \dots, X_n]$ such that

$$\varphi(a_1, \dots, a_n) = (F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)),$$

for all $P(a_1, \dots, a_n) \in V$.

<u>e.g.</u>: Let $V = V(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2(K)$.
The rational parametrization considered in class 2

$$\begin{aligned} \varphi : \mathbb{A}^1(K) &\longrightarrow V \\ t &\longmapsto (t^2 - 1, t(t^2 - 1)) \end{aligned}$$

is an example of morphism between $\mathbb{A}^1$ and $V$.

Note that $\varphi$ is not a bijection between $\mathbb{A}^1(K)$ and $V$, since $\varphi(1) = \varphi(-1) = (0,0)$.

<u>Remark</u>: The image of a morphism is not necessarily an algebraic set.

For instance, the image of the following morphism:

$$\varphi: \mathbb{A}^2(K) \longrightarrow \mathbb{A}^2(K)$$
$$(x,y) \longmapsto (x, xy)$$

is the set $\{(x,y) : x \neq 0\} \cup \{(0,0)\}$ which is not an algebraic set.

Now, to each morphism $\varphi: V \longrightarrow W$ we can associate a $K$-homomorphism between $K[W]$ and $K[V]$ in the following way:

$$\varphi^*: K[W] \longrightarrow K[V]$$
$$\overline{F} \longmapsto \overline{F \circ \varphi}.$$

Note that $\varphi^*$ is well-defined. Indeed, let $F, G \in K[X_1, \dots, X_m]$ such that $\overline{F} = \overline{G}$, i.e. $F - G \in I(W)$.

For all $P \in V$, we have

$$(\varphi^*(\overline{F}) - \varphi^*(\overline{G}))(P) = \varphi^*(\overline{F-G})(P) = ((F-G)\circ\varphi)(P) = \underbrace{(F-G)}_{\in I(W)}(\underbrace{\varphi(P)}_{\in W}) = 0 \Rightarrow$$
$$\Rightarrow \varphi^*(\overline{F}) = \varphi^*(\overline{G}).$$

Vice versa, let $\alpha: K[W] \longrightarrow K[V]$ be a $K$-homomorphism.

The map $\alpha$ is completely described by the image of the generators of $K[W]$:

$$\alpha: K[W] \longrightarrow K[V]$$
$$\overline{X_1} \longmapsto \overline{F_1}$$
$$\vdots$$
$$\overline{X_m} \longmapsto \overline{F_m}$$

If we consider the morphism:

$$\varphi: \mathbb{A}^n(K) \longrightarrow \mathbb{A}^m(K)$$
$$P \longmapsto (F_1(P), \dots, F_m(P)),$$

then it is not difficult to show that $\varphi(V) \subseteq W$.

Therefore $\varphi|_V : V \to W$ is a morphism between $V$ and $W$ and we have $(\varphi|_V)^* = \alpha$.

To sum up, we have the following result:

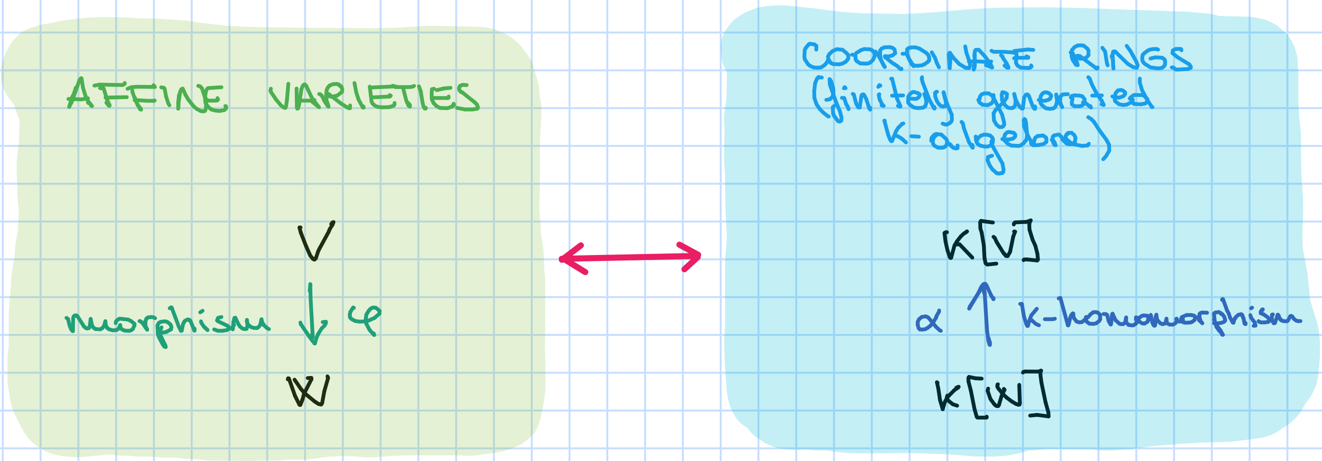
<u>Theorem</u>: The following hold:

1) Every morphism $\varphi: V \to W$ of affine varieties induces a $K$-homomorphism:
$$\varphi^* : K[W] \to K[V]$$
$$F \longmapsto F \circ \varphi$$

2) Every $K$-homomorphism $\alpha: K[W] \to K[V]$ induces a morphism
$$\varphi : V \to W$$
such that $\varphi^* = \alpha$.

3) If $\varphi_1 : V \to W$, $\varphi_2 : W \to Z$ are morphisms of affine varieties, then
$$(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^* .$$

<u>Corollary</u>: The category of affine varieties with morphisms and the category of coordinate rings (i.e. affine $K$-algebras) are contravariantly equivalent).




<u>Def</u>: Let $V \subseteq \mathbb{A}^n(K)$, $W \subseteq \mathbb{A}^m(K)$ be affine varieties. A morphism $\varphi: V \to W$ is an isomorphism if there is a morphism $\psi: W \to V$ such that
$$\psi \circ \varphi = id_V \quad \text{and} \quad \varphi \circ \psi = id_W$$
In this case $V$ and $W$ are said to be isomorphic.

**Remark**: An isomorphism $\varphi: V \to W$ is a bijective morphism with inverse a morphism.

**Corollary**: Two affine varieties are isomorphic if and only if their coordinate rings are isomorphic:

$$V \approx W \iff K[V] \cong K[W].$$

**e.g.**: Any affine change of coordinates on $\mathbb{A}^n(K)$ induces an isomorphism of any variety $V \subseteq \mathbb{A}^n(K)$ with itself.

Indeed an affine change of coordinates is a bijective morphism:

$$T: \mathbb{A}^n(K) \longrightarrow \mathbb{A}^n(K)$$
$$P \longmapsto (F_1(P), \dots, F_n(P)),$$

with $F_i \in K[X_1, \dots, X_n]$ and $\deg(F_i) = 1$.

Recall that $T$ can be written as a composition of a translation and an invertible linear map.