Bridge - MGF 3301 - Section 001

Homework 5

Instructions: Solve the following exercises in a **separate sheet of paper**. Be tidy and organized! You can work on the exercises with your friends (or enemies!) but the final editing has to be yours. This homework has to be returned **by Wednesday February 26 at 9:30 am**. The total number for this homework is 110 (there are 10 extra points). The grade you will receive for this homework will count as a part of *Homework* component of the total grade (15%).

Ex 1. [**40 points total**] Prove by contrapositive the following claims (please, write down the contrapositive for each statement first).

Claim 1 (20 points) *Let n be an integer. If $n^2 - 6n + 5$ is even, then n is odd.*

Claim 2 (20 points) *Let a, b, c be positive real numbers. If $ab = c$ then $a \leq \sqrt{c}$ or $b \leq \sqrt{c}$.*

Ex 2. [**max 35 points**] Prove by contradiction the following claims. In each proof highlight what is the contradiction (i.e. identify the proposition Q such that you have $Q \wedge (\sim Q)$).

Claim 1 (25 points) *The sum of a rational number and an irrational number is irrational.* (Recall that x is said to be a *rational number* if there exist integers a and b, with $b \neq 0$ such that $x = \frac{a}{b}$).

Claim 2 (25 points) *There is no smallest rational number strictly greater than 0.*

Ex 3. [**max 35 points**] Consider the following definitions:

Definition

Let a and b be integers. A **linear combination** of a and b is an expression of the form

$$ax + by,$$

where x and y are also integers. Note that a linear combination of a and b is also an integer.

Definition

Given two integers a and b we say that a **divides** b, and we write $a|b$, if there exists an integer k such that

$$b = ka.$$

Moreover, we write $a \nmid b$ if a does not divide b.

For each proof state clearly which technique you used (direct proof, proof by contrapositive, proof by contradiction). Even if you are not able to prove some of the following claims, you can still use them in the proof of the following ones, if needed.

(a) (10 points) Given the above definition, is it true that $a|0$ for all a in $\mathbb{Z}$? Is it true that $0|a$ for all a in $\mathbb{Z}$? Is it true that $a|a$ for all a in $\mathbb{Z}$? Explain your answers.

(b) (10 points) Prove that if a and b are two integers such that $b \neq 0$ and $a|b$, then $|a| \leq |b|$.

(c) (10 points) Prove that if a, b and c are three integers such that $c|a$ and $c|b$ then c divides any linear combination of a and b.

(d) (10 points) Let a be a natural number and b be an integer. If $a|(b+1)$ and $a|(b-1)$, then $a = 1$ or $a = 2$. (*Hint*: you may use a clever linear combination...)

(e) (10 points) Prove that if a and b are two integers with $a \geq 2$, then $a \nmid b$ or $a \nmid b + 1$.