RELATIONS (Section 3.1)

Recall: Let A, B be sets. Then the product of A and B (cross product or cartesian product) is the set:

$$A \times B := \{ (a,b) \,,\; a \in A,\, b \in B \}.$$

Example

- $A = \{1, 2, 3, 4\}$
  $B = \{a, b, c\}$

  $$A \times B = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c), (4,a), (4,b), (4,c) \}$$

- $B = \{a, b, c\}$, $B = \{a, b, c\}$

  $$B \times B = \{ (a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c) \}$$

Example

$X = \{$ students in Dr Iezzi's bridge class $\}$

$Y = \{1, 2, 3, \dots, 100\}$

Let $x \in X$ and $y \in Y$. We say that $\underbrace{x \text{ is in relation with } y}_{x \sim y}$ if x is y years old.

Example:

Alexy $\sim$ 18

Mary $\sim$ 19

Sebastian $\sim$ 21

Ismary $\sim$ 24

Example : Alexy ~ 18
Mary ~ 19
Sebastian ~ 21
Ismony ~ 24

So I can consider the ordered pairs:

$R = \{(\text{Alexy}, 18), (\text{Mary}, 19), (\text{Sebastian}, 21), (\text{Ismony}, 24), \dots\} \subsetneq X \times Y$

this subset defines a relation from X to Y

30 elements

300 elements

Example 2

$A = \{0, 1, 2, 3, 4\}$
$B = \{0, 1, 2, 3, 4, 5, 6\}$

$A \times B$: 35 elements

Let $a \in A$, $b \in B$. We say that
$a \sim b \iff a \mid b$. ($\exists\, k \in \mathbb{Z}$ s.t. $b = ka$)

$R = \{\underbrace{(a,b) \in A \times B}\ :\ a \sim b\} \subseteq A \times B$

$R = \{(0,0), (1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,0), (2,2), (2,4), (2,6), (3,0), (3,3), (3,6), (4,0), (4,4)\}$

$|R| = \#R = 17$.

Remark : For $a \in A$ and $b \in B$ we identify the the notion $a \sim b$ with the ordered pair $(a,b) \in A \times B$.

Def: Let A and B be sets.

R is a relation from A to B if R is a subset of $A \times B$.

A relation from A to A is called a relation on A.

If $(a,b) \in R$ we say that a is R-related to b and we write:

$$a \sim_R b$$

If $(a,b) \notin R$ we write $a \not\sim_R b$

Example: $A = \{1,2,3,4,5\}$

$B = \{a,b,c\}$

$A \times B$ has 15 elements

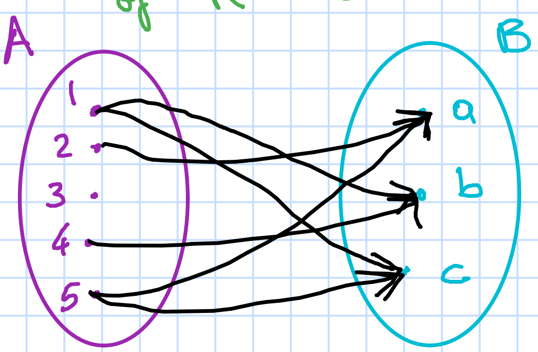
$R = \{(1,b),(1,c),(2,a),(4,b),(5,a),(5,c)\}$

Geometrically

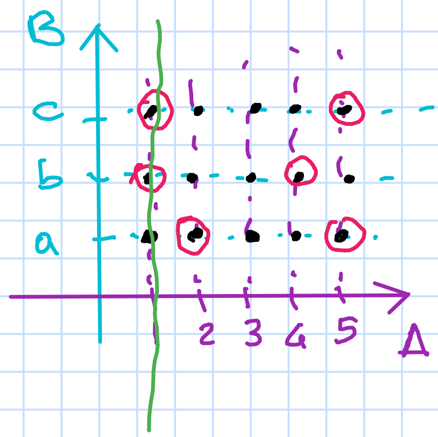
This is not a function because 1 has two different outputs.

We will see that all functions are relations, but there are relations which are not functions!

Arrow diagram of R



Graph of R



it is not a function because it does not pass the vertical line test

Example: $A = \{1, 2, 3, 4, 5\}$

Consider the following relation on A:

$R = \{(1,2), (1,3), (2,2), (2,5), (3,1), (5,4)\} \subseteq A \times A$

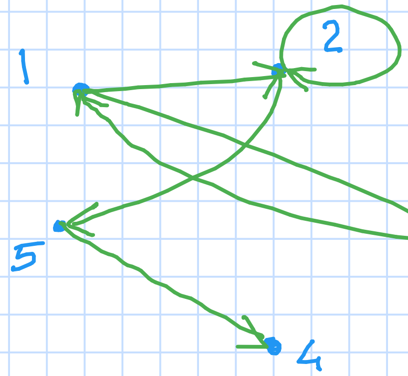
→ arrow diagram

→ graph notation

→ directed graph (digraph)

vertices: elements of A

edges (oriented):



Def: For any set A the identity relation on A is the set:

$$I_A = \{(a,a) : a \in A\} \subseteq A \times A$$

example: $A = \mathbb{R}$, $I_{\mathbb{R}} = \{(x, x) : x \in \mathbb{R}\}$

↑ y-coordinate

$y = x$