EQUIVALENCE RELATIONS (Sec. 3.2)

Properties of relations

Def: Let R be a relation on A ($R \subseteq A \times A$).

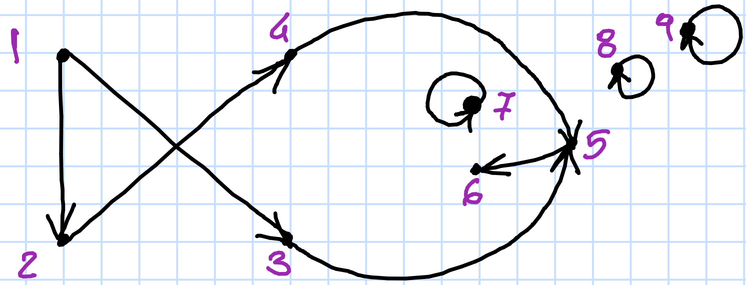
- R is said to be <u>reflexive</u> if $(x,x) \in R$ $\forall x \in A$.

  (equivalently if $I_A = \{(x,x) : x \in A\} \subseteq R$).

- R is said to be <u>symmetric</u> if $\forall x, y \in A$, $(x,y) \in R \Rightarrow (y,x) \in R$.

- R is said to be <u>transitive</u> if $\forall x, y, z \in A$, $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$.

Example 1: Consider the relation associated to the following digraph:



$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$R = \{(1,2), (1,3), (2,4), (3,5), (5,6), (4,5), (7,7), (8,8), (9,9), (10,10)\}$$

- Reflexive? No, because $(1,1) \notin R$
- Symmetric? No, because $(\underset{x}{1},\underset{y}{3}) \in R$, but $(\underset{y}{3},\underset{x}{1}) \notin R$
- Transitive? No, because $(\underset{x}{4},\underset{y}{5}) \in R$, $(\underset{y}{5},\underset{z}{6}) \in R$, but $(\underset{x}{4},\underset{z}{6}) \notin R$.

Example 2 : $A = \mathbb{R}$. Let $x, y \in \mathbb{R}$

$$x \sim y \iff x < y.$$

$$R = \{(x,y) \in A \times A : x \sim y\} \subseteq A \times A$$

- Reflexive? No, because $1 \not\sim 1$ ($1 < 1$ is false)
- Symmetric? No, because $1 \sim 2$ ($1 < 2$) but $2 \not\sim 1$ ($2 < 1$ is false)
- Transitive? Yes.

  Let $x, y, z \in \mathbb{R}$ s.t. $x \sim y$ and $y \sim z$.

  $\xRightarrow{\text{def of relation}}$ $x < y$ and $y < z \Rightarrow x < y < z \Rightarrow$

  $\Rightarrow x < z \Rightarrow x \sim z$.

Example 3: $A = \mathbb{Z}$. Let $x, y \in \mathbb{Z}$

$$x \sim y \iff xy \text{ is odd} \iff x, y \text{ are both odd}$$

- Reflexive? No, for instance $2 \not\sim 2$ (because $2 \cdot 2 = 4$ is even)
- Symmetric? Yes, because the product of integers is commutative.

  If $x \sim y \Rightarrow xy$ is odd $\Rightarrow$

  $\xRightarrow{\text{product commutative } xy = yx}$ yx is odd $\Rightarrow y \sim x$.

- Transitive? Yes.

  Let $x, y, z \in \mathbb{Z}$. Assume that $x \sim y$ and $y \sim z \Rightarrow xy$ is odd and yz is odd $\Rightarrow$ $\Rightarrow x, y, z$ are odd $\Rightarrow xz$ is odd $\Rightarrow x \sim z$.

Example 4: Let A be a non-empty set. Consider the following relation on $\mathcal{P}(A)$.

<u>Recall</u>: $\mathcal{P}(A) = \{B : B \subseteq A\}$

Let $B, C \in \mathcal{P}(A)$. We say

$$B \sim C \iff B \subseteq C.$$

$$R = \{(B,C) \in \mathcal{P}(A) \times \mathcal{P}(A) : \underbrace{B \subseteq C}_{B \sim C}\} \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$$

- Reflexive? Yes, indeed $\forall B \in \mathcal{P}(A)$, $B \sim B$ $(B \subseteq B)$
- Symmetric? No, because $\emptyset, A \in \mathcal{P}(A)$ and $\underbrace{\emptyset \subseteq A}_{\emptyset \sim A}$ but $\underbrace{A \not\subseteq \emptyset}_{A \not\sim \emptyset}$
- Transitive? Yes. Indeed let $B, C, D \in \mathcal{P}(A)$. If $B \sim C$ and $C \sim D \Rightarrow$
  $\Rightarrow B \subseteq C$ and $C \subseteq D \Rightarrow$
  $\Rightarrow B \subseteq C \subseteq D \Rightarrow B \subseteq D \Rightarrow B \sim D$.

For instance, let $A = \{1,2\}$. Then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

$$R = \{(B,C) \in \mathcal{P}(A) \times \mathcal{P}(A) : B \subseteq C\} =$$

$$= \{\underbrace{\boxed{(\emptyset, \{1\})}}_{\emptyset \subseteq \{1\}}{}^{1*}, (\emptyset, \{2\}), (\emptyset, \{1,2\})^{*3}, \underbrace{\cancel{(\{1\}, \emptyset)}}_{\{1\} \not\subseteq \emptyset}, \boxed{(\emptyset,\emptyset)},$$

$$\boxed{(\{1\},\{1\})}, \underbrace{(\{1\}, \{1,2\})^{*2}}_{\{1\} \subseteq \{1,2\}}, \boxed{(\{2\},\{2\})}, (\{2\}, \{1,2\}),$$

$$\boxed{(\{1,2\}, \{1,2\})}\}$$

Def: A relation R on a set A is an <u>equivalence relation</u> on A if:

- R is reflexive
- R is symmetric
- R is transitive

Example: $I_A = \{(x,x) : x \in A\} = \{(x,y) \in A \times A : x = y\}$

identity

Two elements $x, y \in A$ are in relation if and only if $x = y$.

Remark: If R is an equivalence relation on A, then $I_A \subseteq R$.

Example: Consider $X = \{$students in Dr Iezzi's class$\}$

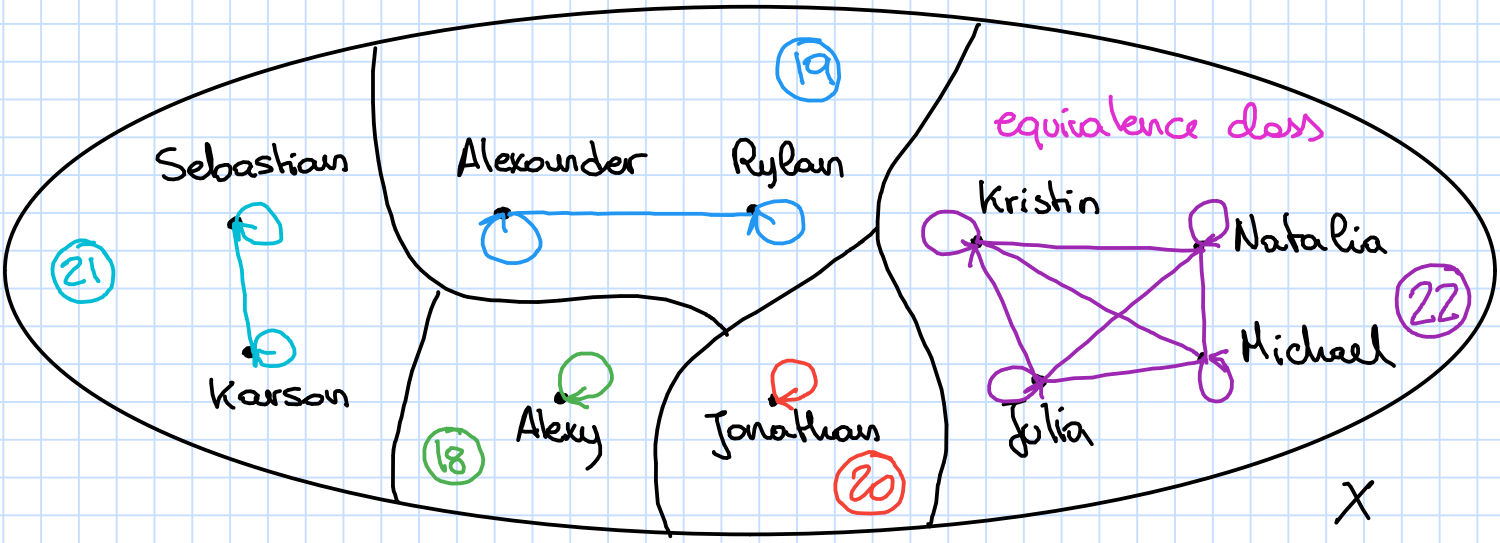
Let $x, y \in X$. We say:

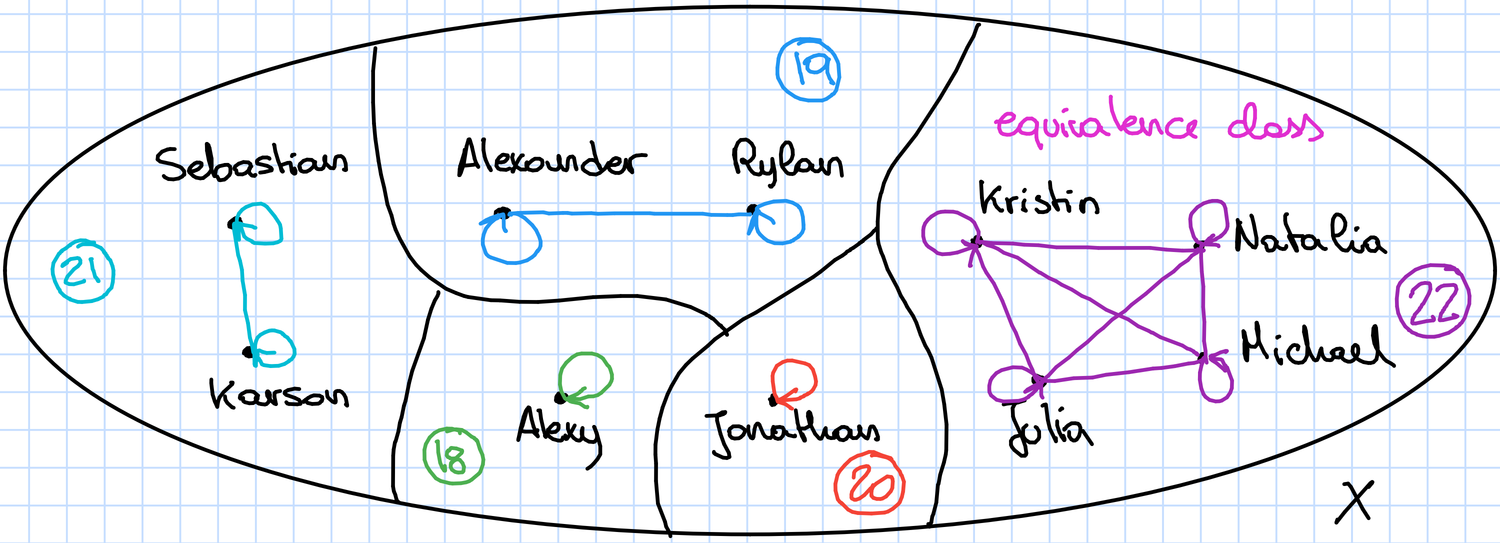
$$x \sim y \iff x \text{ and } y \text{ have the same age.}$$

$$R = \{(x,y) \in X \times X : x \sim y\}$$

$X = \{$Natalia (22), Jonathan (20), Kristin (22), Alexy (18), Rylan (19),

Julia (22), Alexander (19), Michael (22), Karson (21), Sebastian (21)$\}$





$\forall x \in X$, let us consider the subset of X:

$$\overline{x} = \{ y \in X : x \sim y \} \subseteq X$$

$\overline{\text{Kristin}} = \{ y \in X : \text{Kristin} \sim y \} =$

$= \{ y \in X : \text{Kristin and } y \text{ have the same age} \} =$

$= \{ \text{Kristin}, \text{Natalie}, \text{Michael}, \text{Julie} \} = \overline{\text{Julie}}$

representative

$\overline{\text{Jonathan}} = \{ \text{Jonathan} \}$

$\overline{\text{Alexy}} = \{ \text{Alexy} \}$

$\overline{\text{Sebastian}} = \{ \text{Karson}, \text{Sebastian} \}$

$\overline{\text{Alexander}} = \{ \text{Alexander}, \text{Rylan} \}$

partition of X

$X = \overline{\text{Kristin}} \cup \overline{\text{Jonathan}} \cup \overline{\text{Alexy}} \cup \overline{\text{Sebastian}} \cup \overline{\text{Alexander}}$

X/R (X modulo R):

$X/R = \{ \overline{\text{Kristin}}, \overline{\text{Jonathan}}, \overline{\text{Alexy}}, \overline{\text{Sebastian}}, \overline{\text{Alexander}} \}$

Def: Let R be an <u>equivalence relation</u> on a set A.
For $x \in A$, the <u>equivalence class</u> of x modulo R (or simply x mod R) is the subset of A:

$$\bar{x} = \{ y \in A : \underset{(x,y) \in R}{x \sim y} \} \subseteq A$$

Each element of $\bar{x}$ is called a <u>representative</u> of the class $\bar{x}$.
The set:

this is a subset of A

$$A/R = \{ \bar{x} : x \in A \} \not\subseteq A$$

of all equivalence classes is called A <u>modulo R</u>.