EQUIVALENCE RELATIONS (3.2)
PARTITIONS (3.3)

Def: Let R be an equivalence relation (reflexive, symmetric and transitive) on a set A ($R \subseteq A \times A$).

For $x \in A$, the equivalence class of x modulo R is the set:

$$\bar{x} = [x]_R := \{ y \in A : \underbrace{(x,y)}_{\overset{\shortparallel}{(y,x)}} \in R \} \subseteq A$$

(by symmetry).

Each element of $\bar{x}$ is called a representative of the class $\bar{x}$.

The set:

$$A/R := \{ \bar{x} : x \in A \} \not\subseteq A$$

of all equivalence classes is called A modulo R.

Example 1: $R = \{ (x,y) \in \underbrace{\mathbb{Z} \times \mathbb{Z}}_{\mathbb{Z}^2} : x+y \text{ is even} \}$.

This is an equivalence relation.

- reflexive: $\forall x \in \mathbb{Z}$, $x + x = 2x$ is even $\Rightarrow (x,x) \in R$.
- symmetric: $\forall x,y \in \mathbb{Z}$ if $(x,y) \in R$, then $x+y$ is even $\Rightarrow$ $y+x$ is even (addition is commutative) $\Rightarrow (y,x) \in R$.
- transitive: $\forall x,y,z \in \mathbb{Z}$, let us assume $(x,y),(y,z) \in R \Rightarrow x+y$ is even and $y+z$ is even. We have:

$$x+z = \underbrace{x+y}_{\text{even}} + \underbrace{y+z}_{\text{even}} - \underbrace{2y}_{\text{even}} = 2h + 2k - 2y =$$

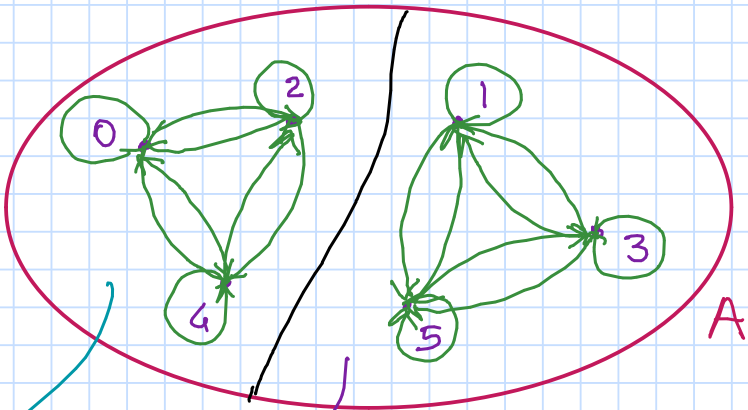
$\exists h,k \in \mathbb{Z}$

$$= 2(h+k-y) \text{ is even} \Rightarrow (x,z) \in R.$$

Let's replace for a moment $\mathbb{Z}$ with $A = \{0,1,2,3,4,5\}$.

In this case $R = \{(x,y) \in A^2 : x+y \text{ is even}\}$

$$R = \{(0,0), (0,2), (0,4), (1,1), (1,3), (1,5), (2,0), (2,2), (2,4), (3,1), (3,3), (3,5), (4,0), (4,2), (4,4), (5,1), (5,3), (5,5)\}$$



$\bar{0} = \{0,2,4\} = \bar{2} = \bar{4}$

$\bar{1} = \{1,3,5\} = \bar{3} = \bar{5}$

$\Big\} \Rightarrow A/R = \{\bar{0}, \bar{1}\} = \{\{0,2,4\}, \{1,3,5\}\}.$

Let's go back to $\mathbb{Z}$.

this is not hard to prove

$(x,y) \in R \iff x+y$ is even $\iff$ x,y are both even or x,y are both odd.

So we have:

$$\bar{0} = \{y \in \mathbb{Z} : (0,y) \in R\} = \{y \in \mathbb{Z} : 0+y \text{ is even}\} = \{y \in \mathbb{Z} : y \text{ is even}\}$$

$$\bar{1} = \{y \in \mathbb{Z} : (1,y) \in R\} = \{y \in \mathbb{Z} : 1+y \text{ is even}\} = \{y \in \mathbb{Z} : y \text{ is odd}\}$$

$\bar{0} \cup \bar{1} = \{\text{even integers}\} \cup \{\text{odd integers}\} = \mathbb{Z}$

$\mathbb{Z}/R = \{\bar{0}, \bar{1}\}$

Example 2: $R = \{(x,y) \in \mathbb{R}^2 : x^2 = y^2\}$.

- reflexive
- symmetric
- transitive

$\Big\} \Rightarrow R$ is an equivalence relation.

Equivalence classes.

$\bar{0} = \{y \in \mathbb{R} : (0,y) \in R\} = \{y \in \mathbb{R} : 0 = y^2\} =$
$= \{0\}$

$\bar{1} = \{y \in \mathbb{R} : (1,y) \in R\} = \{y \in \mathbb{R} : 1 = y^2\} =$
$= \{1, -1\}$

$\forall \alpha \in \mathbb{R}, \alpha \neq 0$:
$\bar{\alpha} = \{\alpha, -\alpha\}$.

$\mathbb{R}/R = \{\bar{x} : x \in \mathbb{R}\} = \{\bar{x} : x \in \mathbb{R}, x \geq 0\}$

for each class I can find a representative ≥ 0

Theorem: Let R be an equivalence relation on a non-empty set A. $\forall x, y \in A$

(a) $x \in \bar{x}$ and $\bar{x} \subseteq A$

(b) $(x,y) \in R \iff \bar{x} = \bar{y}$

(c) $(x,y) \notin R \iff \bar{x} \cap \bar{y} = \emptyset$.

If $x, y \in A$ then
- either $\bar{x} = \bar{y}$
- or $\bar{x} \cap \bar{y} = \emptyset$

Proof

(a) $x \in \bar{x}$ because $(x,x) \in R$, since R is reflexive.
$\bar{x} \subseteq A$ by definition.

(b) $\Rightarrow)$ $(x,y) \in R \Rightarrow \bar{x} = \bar{y}$.

Assume that $(x,y) \in R \Rightarrow (y,x) \in R$ (R is symmetric). We want to prove that $\bar{x} = \bar{y}$

($\subseteq$) Let $z \in \bar{x} \Rightarrow (x,z) \in R$. We know also that
$(y,x) \in R \Rightarrow (y,z) \in R$ (transitivity) $\Rightarrow z \in \bar{y}$.

($\supseteq$) Let $z \in \bar{y} \Rightarrow (y,z) \in R$. We know that $(x,y) \in R \Rightarrow (x,z) \in R \Rightarrow z \in \bar{x}$

$\Leftarrow)$ $\bar{x} = \bar{y} \Rightarrow (x,y) \in R$.

Assume that $\bar{x} = \bar{y} \Rightarrow y \in \bar{y}$ (because of (a))

and $\bar{y} = \bar{x} \Rightarrow y \in \bar{x} \Rightarrow (x,y) \in R$.

(c) $\Rightarrow)$ $(x,y) \notin R \Rightarrow \bar{x} \cap \bar{y} = \emptyset$.

Assume that $\underbrace{(x,y) \notin R}_{Q}$. Assume also, to the contrary, that $\bar{x} \cap \bar{y} \neq \emptyset \Rightarrow \exists\, z \in \bar{x} \cap \bar{y}$,

$\Rightarrow z \in \bar{x}$ and $z \in \bar{y} \Rightarrow (x,z) \in R$ and

$\underbrace{(y,z) \in R}_{\Downarrow\ (z,y) \in R} \Rightarrow \underbrace{(x,y) \in R}_{\sim Q}$ (by transitivity) $\lightning$

So $\bar{x} \cap \bar{y} = \emptyset$

$\Leftarrow)$ $\bar{x} \cap \bar{y} = \emptyset \Rightarrow (x,y) \notin R$

Assume that $\overbrace{\bar{x} \cap \bar{y} = \emptyset}^{Q}$. Assume, to the contrary, that $(x,y) \in R \Rightarrow y \in \bar{y}$ (by (a))

and $y \in \bar{x} \Rightarrow y \in \bar{x} \cap \bar{y} \Rightarrow \underbrace{\bar{x} \cap \bar{y} \neq \emptyset}_{\sim Q}$ $\lightning$

So $(x,y) \notin R$.

Because of the previous theorem:

$A/R = \{\bar{x} : x \in A\}$ is a partition of A

Def: Let A be a non-empty set.

A partition $\mathcal{P}$ of A is a set of subsets of A such that:

(a) If $B \in \mathcal{P} \Rightarrow B \neq \emptyset$. (each element of $\mathcal{P}$ is non-empty)

(b) If $B \in \mathcal{P}$ and $C \in \mathcal{P} \Rightarrow \underbrace{B = C \text{ or } B \cap C = \emptyset}_{\text{sets in } \mathcal{P} \text{ are pairwise disjoint}}$

(c) $\bigcup_{B \in \mathcal{P}} B = A$

A partition of a set A is a $\overset{(b)}{\text{pairwise disjoint}}$ family of $\underbrace{\text{nonempty}}_{(a)}$ subsets of A whose $\underbrace{\text{union is A}}_{(c)}$

Theorem: If R is an equivalence relation on a nonempty set A, then A/R is a partition of A.

Proof:

$A/R = \{\overline{x} : x \in A\}$

(a) $\forall x \in A$, $\overline{x} \neq \emptyset$ since $x \in \overline{x}$.

(b) $\forall x, y \in A$, either $\overline{x} = \overline{y}$ or $\overline{x} \cap \overline{y} = \emptyset$ (by previous theorem). So sets in A/R are pairwise disjoint.

(c) We have to prove that $\bigcup_{x \in A} \overline{x} = A$.

$\subseteq$: Let $y \in \bigcup_{x \in A} \overline{x} \Rightarrow \exists\, x \in A$ s.t. $y \in \overline{x} \subseteq A$
$\Rightarrow y \in A$.

$\supseteq$: Let $y \in A \Rightarrow y \in \overline{y} \subseteq \bigcup_{x \in A} \overline{x}$.

We can also define an equivalence relation on a nonempty set A starting from a partition $\mathcal{P}$ of A.

$$R = \{(x, y) \in A : \exists\, B \in \mathcal{P} \text{ s.t. } x, y \in B\}$$

- reflexive: Since $A = \bigcup_{B \in \mathcal{P}} B$, $\forall x \in A$, $x \in \bigcup_{B \in \mathcal{P}} B \Rightarrow$
  $\Rightarrow \exists\, B \in \mathcal{P}$ s.t. $x \in B \Rightarrow \exists\, B \in \mathcal{P}$ s.t. $x, x \in B$
  $\Rightarrow (x, x) \in R$.

- symmetric: $\forall x, y \in A$ s.t. $(x, y) \in R \Rightarrow \exists\, B \in \mathcal{P}$ s.t.
  $x, y \in B \Rightarrow \exists\, B \in \mathcal{P}$ s.t. $y, x \in B \Rightarrow (y, x) \in R$.

- transitive: $\forall x, y, z \in A$ s.t. $(x, y) \in R$ and $(y, z) \in R \Rightarrow$
  $\Rightarrow \exists\, B, C \in \mathcal{P}$ s.t. $x, y \in B$ and $y, z \in C \Rightarrow$
  $\Rightarrow y \in B \cap C \Rightarrow \overset{\mathcal{P} \text{ is a partition}}{B \cap C \neq \emptyset} \Rightarrow B = C \Rightarrow$
  $\Rightarrow \exists\, B \in \mathcal{P}$ s.t. $x, z \in B \Rightarrow (x, z) \in R$.