Bridge - MGF 3301 - Section 001

Quiz 6 - Solution

04/08/2020

Exercise 1
(12 points)

Consider the following relation on $\mathbb{Z}$:

$$R = \{(a, b) \in \mathbb{Z}^2 : 3 \mid (a - b)\}.$$

(a) (2 points) Which ordered pairs among the following belongs to R? Select all that apply.

- ■ (23,17)
- ■ (17,23)
- □ (18,17)
- ■ (17,17)

(b) (2 points) Prove that R is reflexive on $\mathbb{Z}$.

Solution

For all $a \in \mathbb{Z}$ we have that $a - a = 0$ and $3 \mid 0$. So $(a, a) \in R$, $\forall\, a \in \mathbb{Z}$, which implies that R is reflexive.

(c) (2 points) Prove that R is symmetric.

Solution

Let $a, b \in \mathbb{Z}$ such that $(a, b) \in R$, i.e. $3 \mid (a - b)$. Then, by definition, there exists $k \in \mathbb{Z}$ such that $a - b = 3k$. Then $b - a = 3 \cdot (-k)$, which implies that $3 \mid (b - a)$. Hence $(b, a) \in R$, which proves that R is symmetric.

(d) (2 points) Prove that R is transitive.

Solution

Let $a, b, c \in \mathbb{Z}$. Assume that $(a, b), (b, c) \in R$, i.e. $3 \mid (a - b)$ and $3 \mid (b - c)$. Then, by definition, there exist $k, h \in \mathbb{Z}$ such that $a - b = 3k$ and $b - h = 3h$. We have:

$$a - c = a - b + b - c = 3k + 3h = 3(k + h).$$

Therefore $3 \mid (a - c)$, which implies $(a, c) \in R$, proving that R is transitive.

Recall that for $a \in \mathbb{Z}$, we denote $\overline{a} := \{b \in \mathbb{Z} : (a, b) \in R\}$.

(e) (2 points) Prove that, for the relation R defined previously, we have $\overline{0} = 3\mathbb{Z}$.

Solution

We have

$$\begin{aligned}\overline{0} &= \{b \in \mathbb{Z} : (0, b) \in R\} = \\ &= \{b \in \mathbb{Z} : (b, 0) \in R\} = \\ &= \{b \in \mathbb{Z} : 3 \mid (b - 0)\} = \\ &= \{b \in \mathbb{Z} : 3 \mid b\} = \\ &= \{b \in \mathbb{Z} : \exists\, k \in \mathbb{Z} \text{ such that } b = 3k\} = \\ &= 3\mathbb{Z}.\end{aligned}$$

(f) (2 points) Prove that if $x \in \overline{1}$ then $x = 3k + 1$, for some integer k.

Solution

Let $x \in \overline{1}$. Then $(x, 1) \in R$, i.e. $3 \mid (x - 1)$. Hence, there exists $k \in \mathbb{Z}$ such that $x - 1 = 3k$, or equivalently $x = 3k + 1$.