TEST 2 - STUDY GUIDE

Bridge - MGF 3301 - Section 001

When? The first test will take place on **Wednesday March 11 at 9:30 am** in CMC 118.

Topics: **Sections 1.4, 1.5, 1.6, 2.1, 2.2, 2.4** of the textbook *A Transition to Advanced Mathematics*, by Smith, Eggen & St. Andre, 8th edition.

Office hours:

- Monday March 9: 11am-12pm
- Tuesday March 10: 4-5:30pm.

For the second test, you need to be able to:

- Prove a claim directly, by contrapositive or by contradiction. Claims may be conditional sentences, biconditional sentences or of other form, and may contain quantifiers (see Homework 5).
- Disprove a statement with a counterexample.
- Translate a set in *roster notation* into a set in *set-builder notation* (see Exercise 1 of Homework 6)
- Use correctly the symbols $\in, \notin, \subseteq, \nsubseteq, \subsetneq, \varnothing$ (see Exercise 2 of Homework 6 and Exercise 2 of Homework 7).
- Build examples of sets that verify specific properties (see Exercise 3 of Homework 6).
- Prove directly that $A \subseteq B$ or $A = B$.
- Prove statements about set theory (see Exercise 3 of Homework 7).
- For two given sets A and B, compute $A \cup B$, $A \cap B$, $A \setminus B$, $\mathcal{P}(A)$, $\overline{A}$ (see Exercise 1 of Homework 7).
- Prove by induction that a statement is true for all natural numbers (see Exercise 4 of Homework 7).

Moreover, you have to:

- Know all the definitions that appear in the next page.
- Know and <u>understand</u> the proof of all the claims that appear in the next pages (their structure may be useful for some of the proofs of the claims that you will find in the test). Important: you do not have to learn them by heart, but understand the structure and the logic steps for going from the assumption(s) to the conclusion(s).

Review:

- Quizzes 4, 5 (and their solutions).
- Homework 5, 6, 7.
- Read again all the notes and/or Sections 1.4, 1.5, 1.6, 2.1, 2.2, 2.4 of the textbook.

Definitions

- A set A is a **subset** of a set B if every element of A is an element of B. In this case we write $A \subseteq B$. So we have
$$\text{“}A \subseteq B\text{”} \Leftrightarrow \text{“}x \in A \Rightarrow x \in B\text{”}.$$
- We say that a set A is a **proper subset** of a set B if $A \subseteq B$ and $A \neq B$. In this case we write $A \subsetneq B$.
- The **power set** of a set A is the set whose elements are the subsets of A. It is denoted $\mathcal{P}(A)$. So we have:
$$\mathcal{P}(A) := \{B : B \subseteq A\}.$$
- Given two sets A and B, the **union** $A \cup B$ is the set
$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$
- Given two sets A and B, the **intersection** $A \cap B$ is the set
$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$
- Given two sets A and B, the **difference** $A \setminus B$ is the set
$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}.$$
- Two sets A and B are **disjoint** if $A \cap B = \varnothing$.
- Let U be the universe and $A \subseteq U$. The **complement** of A is the set
$$\overline{A} = U \setminus A.$$
- An integer n is said to be **even** if $\exists\, k$ in $\mathbb{Z}$ such that $n = 2k$.
- An integer n is said to be **odd** if $\exists\, k$ in $\mathbb{Z}$ such that $n = 2k + 1$.
- Given two integers a and b we say that a **divides** b, and we write $a|b$, if there exists an integer k such that
$$b = ka.$$
Moreover, we write $a \nmid b$ if a does not divide b.
- Let $a \in \mathbb{Z}$. We denote by $a\mathbb{Z}$ the set of multiple integers of a, i.e.
$$a\mathbb{Z} := \{n \in \mathbb{Z} : n = ak,\ k \in \mathbb{Z}\}.$$

Proof: direct and by contrapositive

Claim 1: For all $n \in \mathbb{Z}$, n is even if and only if n^2 is even.

Proof:

$\Rightarrow$) Let $n \in \mathbb{Z}$. Let us assume that n is even. Then, by definition there exists $k \in \mathbb{Z}$ such that $n = 2k$. So $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ is also an even integer.

$\Leftarrow$) We will prove this implication by contrapositive, i.e. we will prove that if n is odd, then n^2 is odd. Let us assume that n is an odd integer. Then, by definition there exists $k \in \mathbb{Z}$ such that $n = 2k+1$. So $n^2 = (2k+1)^2 = 4k^2+2k+1 = 2(2k^2+k)+1$ is also an odd integer. Therefore if n^2 is even then n is even.

Proof: by contradiction

Claim 2: $\sqrt{2}$ is an irrational number.

Proof:

Assume to the contrary that $\sqrt{2}$ is a rational number, i.e. that there exist integers a and b, with $b \neq 0$, such that

$$\sqrt{2} = \frac{a}{b}.$$

Without loss of generality we can also assume that a and b have no common factors (otherwise we can simplify the fraction). We have:

$$\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \overset{b\neq 0}{\Rightarrow} a^2 = 2b^2.$$

So a^2 is even, which implies that a is even (see Claim 1). Then, by definition, there exists $k \in \mathbb{Z}$ such that $a = 2k$. By substituting in $a^2 = 2b^2$ we obtain:

$$(2k)^2 = 2b^2 \Rightarrow 4k^2 = 2b^2 \Rightarrow b^2 = 2k^2 \Rightarrow b^2 \text{ is even } \Rightarrow b \text{ is even }.$$

In conclusion a and b are both even, so 2 is a common factor. This contradicts the assumption that a and b have no common factors. Therefore $\sqrt{2}$ is an irrational number.

Proof: by induction

Claim 3: For all $n \in \mathbb{N}$, the sum of the first n odd natural integers is equal to n^2, i.e.

$$\forall\, n \in \mathbb{N},\ 1 + 3 + \cdots + (2n-1) = n^2$$

Proof:

Let $P(n) =$ "$1 + 3 + \cdots + (2n-1) = n^2$". We will prove the claim by induction.

- Basic step: For $n = 1$, we have $P(1) =$ "$1 = 1^2$", so $P(1)$ is true.
- Inductive step: Let us assume that $P(n)$ is true, i.e. that

  $$1 + 3 + \cdots + (2n-1) = n^2.$$

  We want to show that $P(n+1) =$ "$1 + 3 + \cdots + (2n+1) = (n+1)^2$" is also true. We have:

  $$1 + 3 + \cdots + (2n+1) = \underbrace{1 + 3 + \cdots + (2n-1)}_{=n^2} + (2n+1) =$$
  $$= n^2 + 2n + 1 = (n+1)^2.$$

  Note that we used our inductive hypothesis in the second equality. Therefore $P(n+1)$ is also true.

In conclusion, by induction, $\forall\, n \in \mathbb{N}$, $1 + 3 + \cdots + (2n-1) = n^2$.