Bridge - MGF 3301 - Section 001

TEST 1 - Solution

02/12/2020

Print your name and sign below, and read the instructions. Do not open the test until you are told to do so.

Name:	
U number:	
Signature:	

Instructions

This test contains 6 exercises. The total number of points is 110 (there are 10 bonus points). Calculators are not allowed (and actually not needed).

Put all your answers in the spaces provided on these sheets. The last sheet of the test is blank and may be used for scratch work. More scratch paper is available on request.

Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

Do not write in this table

1		4	
2		5	
3		6	

Exercise 1 (10 points)

Consider the following propositions:

- $P :=$ "An even number can be written in the form $2k + 1$, for some integer k."
- $Q :=$ "A square has 5 sides."
- $R :=$ "There exists a rational number x such that $0 < x < \frac{1}{2}$."

a) [2 points] The **truth value** of P is

☐ TRUE

■ FALSE

b) [2 points] The **truth value** of Q is

☐ TRUE

■ FALSE

c) [2 points] The **truth value** of R is

■ TRUE

☐ FALSE

d) [4 points] For the above propositions P, Q and R, determine the **truth value** of the propositional form

$$(P \wedge (\sim Q)) \Rightarrow (R \vee P).$$

Justify your answer.

Solution

For P False, Q False and R True, we have:

P	Q	R	$\sim Q$	$P \wedge (\sim Q)$	$R \vee P$	$(P \wedge (\sim Q)) \Rightarrow (R \vee P)$
F	F	T	T	F	T	T

Therefore the truth value of $(P \wedge (\sim Q)) \Rightarrow (R \vee P)$ for the above propositions is True.

Exercise 2 (15 points)

Write a **non-trivial denial** (i.e. not of the form *It is not the case that...*, *There exists no...*, *Not all...*, *etc.*) of the following propositions:

a) x *is even and* y *is divisible by 3.*

Solution

- x *is not even or* y *is not divisible by 3.*
- x *is odd or* y *is not divisible by 3.*

b) *The function* f *has a local maximum at* $x = -1$ *or at* $x = 1$.

Solution

The function f *does not have a local maximum at* $x = -1$ *nor at* $x = 1$.

c) $\forall x$ *in* $\mathbb{R}$, x^2 *is a rational number.*

Solution

- $\exists x$ *in* $\mathbb{R}$ *such that* x^2 *is not a rational number.*
- $\exists x$ *in* $\mathbb{R}$ *such that* x^2 *is an irrational number.*

d) $\exists n$ *in* $\mathbb{Z}$ *such that* $n + 1$ *is a prime number.*

Solution

$\forall n$ *in* $\mathbb{Z}$, $n + 1$ *is not a prime number.*

e) $\exists x$ *in* $\mathbb{R}$ *such that* $\forall y$ *in* $\mathbb{R}$ $xy = 0$.

Solution

$\forall x$ *in* $\mathbb{R}$, $\exists y \in \mathbb{R}$ *such that* $xy \neq 0$.

Exercise 3 (33 points)

For x a real number, let $P(x)$ and $Q(x)$ be the following open sentences:

$$P(x) := \text{“}x^2 - 3x + 2 = 0,\text{”} \qquad Q(x) := \text{“}x \geq 0.\text{”}$$

a) [3 points] Determine the **truth set** of $P(x)$. Justify your answer.

Solution

The truth set of $P(x)$ consists of all the real solutions of the equation $x^2 - 3x + 2 = 0$. We have:

$$x^2 - 3x + 2 = 0 \Leftrightarrow (x-1)(x-2) = 0 \Leftrightarrow x = 1 \text{ or } x = 2.$$

Therefore the truth set of $P(x)$ is $\{1, 2\}$.

b) [2 points] Determine the **truth set** of $Q(x)$. (Write it as an interval.)

Solution

The truth set of $Q(x)$ is the interval $[0, \infty)$.

c) [3 points] Determine the **truth set** of $P(x) \vee Q(x)$. Justify your answer.

Solution

The truth set of $P(x) \vee Q(x)$ is given by the union of the truth set of $P(x)$ and the truth set of $Q(x)$, i.e $\{1, 2\} \cup [0, \infty) = [0, \infty)$.

d) [4 points] Determine the **truth value** of the proposition "$\exists!\, x$ in $\mathbb{R}$ such that $P(x) \wedge Q(x)$". Justify your answer.

Solution

The truth set of $P(x) \wedge Q(x)$ is given by the intersection of the truth set of $P(x)$ and the truth set of $Q(x)$, i.e $\{1, 2\} \cap [0, \infty) = \{1, 2\}$. Since it does not contain exactly one value, we have that the proposition "$\exists!\, x$ in $\mathbb{R}$ such that $P(x) \wedge Q(x)$" is False.

e) Consider now the conditional sentence:

$$R(x) := \text{“}P(x) \Rightarrow Q(x)\text{”} = \text{“}\textit{If } x^2 - 3x + 2 = 0 \textit{ then } x \geq 0.\text{”}$$

- [3 points] What is the **truth value** of $R(0)$? Justify your answer.

Solution

The proposition $R(0)$ is True, since for $x = 0$ the antecedent of $R(x)$ is false (we get $2 \neq 0$).

- [3 points] What is the **truth value** of $R(1)$? Justify your answer.

Solution

The proposition $R(1)$ is True, since for $x = 1$ both the antecedent and the consequent of $R(x)$ are true.

- [5 points] What is the **truth value** of "$\forall x$ in $\mathbb{R}$, $R(x)$"? Justify your answer.

Solution

The proposition "$\forall x$ in $\mathbb{R}$, $R(x)$" is True. Indeed, for every real number x such that $x \neq 1$ and $x \neq 2$ we have that the antecedent is false, which implies that the conditional sentence $R(x)$ is true; while for $x = 1$ or $x = 2$ both the antecedent and the consequent of $R(x)$ are true, which implies again that the conditional sentence $R(x)$ is true. Therefore in any case (i.e. for all real number x) we have that $R(x)$ is true.

You could have also argued that "$\forall x$ in $\mathbb{R}$, $R(x)$" is true since the truth set of $P(x)$ is contained in the truth set of $Q(x)$.

- [5 points] Write the **converse** of $R(x)$. Is it true for all x in $\mathbb{R}$? Justify your answer.

Solution

The converse of $R(x)$ is:

$$\text{“}Q(x) \Rightarrow P(x)\text{”} = \text{“}\textit{If } x \geq 0 \textit{ then } x^2 - 3x + 2 = 0.\text{”}$$

This statement is not true for all x since, for instance, for $x = 0$ the antecedent is true, but the consequent is false ($2 \neq 0$).

- [5 points] Write the **contrapositive** of $R(x)$. Is it true for all x in $\mathbb{R}$? Justify your answer.

Solution

The contrapositive of $R(x)$ is:

$$\text{“}{\sim}Q(x) \Rightarrow {\sim}P(x)\text{”} = \text{“}\textit{If } x < 0 \textit{ then } x^2 - 3x + 2 \neq 0.\text{”}$$

This statement is true for all x in $\mathbb{R}$ because the contrapositive of $R(x)$ is equivalent to $R(x)$, and we have already proved that $R(x)$ is true for all x in $\mathbb{R}$.

Exercise 4 (29 points)

a) [9 points] Write the following **definitions**:

- An integer n is said to be *even* if and only if...

Solution

...there exists k in $\mathbb{Z}$ such that $n = 2k$.

- An integer n is said to be *odd* if and only if...

Solution

...there exists k in $\mathbb{Z}$ such that $n = 2k + 1$.

- Let n, m be integers. The integer n is said to be *divisible by* m if and only if...

Solution

...there exists k in $\mathbb{Z}$ such that $n = km$.

b) [10 points] Prove that the following two propositional forms are **equivalent**:

$$P \Rightarrow (Q \wedge R) \quad \text{and} \quad (P \Rightarrow Q) \wedge (P \Rightarrow R).$$

Solution

P	Q	R	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

c) [10 points] **Prove** the following claim:

Let n be an integer. If *n is even* then *$n+1$ is odd and n^2 is divisible by* 4.

Solution

Let us assume that n is an even integer. Then, by definition, there exists k in $\mathbb{Z}$ such that $n = 2k$. Hence, we have that $n + 1 = 2k + 1$ is odd by definition. Moreover $n^2 = (2k)^2 = 4k^2 = 4h$, where $h = k^2$ is an integer. Therefore, by definition, n^2 is divisible by 4.
We conclude that if n is even then $n + 1$ is odd and n^2 is divisible by 4.

Exercise 5 (13 points)

Consider the following definition:

Definition

An integer n is said to be a **perfect square** if there exists an integer k such that $n = k^2$.

a) [3 points] Give **three** different **examples** of integers that are perfect squares.

Solution

The integers $0 = 0^2, 1 = 1^2$ and $4 = 2^2$ are three different examples of perfect squares.

b) [10 points] Use the above definition to **prove** the following claim:

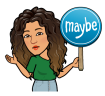
Let n and m be two integers. If *n and m are perfect squares,* then *also their product is a perfect square.*

Solution

Let us assume that n and m are perfect squares. Then, by definition, there exist h and k in $\mathbb{Z}$ such that $n = h^2$ and $m = k^2$. Hence, we have that $nm = h^2k^2 = (hk)^2 = \ell^2$, where $\ell = hk$ is an integer. Therefore nm is also a perfect square.

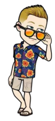
Exercise 6 (10 points) – **Who is telling the truth?**

During the interrogatory of a trial, Anna, Diego and Vanessa made the following statements:



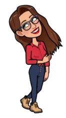
Anna

"Exactly two people (among Diego, Vanessa and me) are lying."



Diego

"Vanessa or I are telling the truth."



Vanessa

"Diego is telling the truth."

In front of these declarations, the trial judge feels the need of consulting a specialist. They call then Andrea, a student in Bridge to Abstract Mathematics, in order to complete the investigation. After thinking for a while, Andrea says with regret to the judge:

Andrea: *"Unfortunately, I do not have enough information in order to uniquely identify all the people who are telling the truth."*

Do you agree with your classmate Andrea? Explain fully and concisely why or why not.

Solution

Yes, I agree with Andrea. Indeed there are three different compatible configurations:

Anna	True
Diego	False
Vanessa	False

Anna	False
Diego	True
Vanessa	True

Anna	False
Diego	False
Vanessa	False

Indeed, if Anna is telling the truth, then necessarily Diego and Vanessa have to lie. So their denials (resp. *"Vanessa and I are lying"* and *"Diego is lying"*) are true, and this does not lead to any contradiction.

Instead, if Anna is lying, then either both Diego and Vanessa are telling the truth (so exactly one among them would be lying), or both Diego and Vanessa are lying (so everybody would be lying). In both cases we do not get any contradiction.

Therefore we need more information in order to uniquely identify all the people who are telling the truth.