Bridge - MGF 3301 - Section 001

TEST 2 - Solution

03/11/2020

Print your name and sign below, and read the instructions. Do not open the test until you are told to do so.

Name:	
U number:	
Signature:	

Instructions

This test contains 7 exercises. The total number of points is 120, but your grade will be the minimum between your score and 110 (you can get up to 10 bonus points). Calculators are not allowed (and actually not needed).

Put all your answers in the spaces provided on these sheets. The last sheet of the test is blank and may be used for scratch work. More scratch paper is available on request.

In each proof you may use a result proved in class, in the homework or in previous exercises of this test. In this case just state clearly which result you are using, but there is no need of proving it again.

Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

Do not write in this table

1		5	
2		6	
3		7	
4		**TOTAL**	

Exercise 1 (27 points)

Consider the following set:

$$A = \{\{\varnothing\}, 3, \{3, 4\}\},$$

(a) [12 points] True or false?

a1) $3 \in A \cap 3\mathbb{Z}$

■ TRUE
☐ FALSE

a2) $\{\{\varnothing\}, 3\} \in A$

☐ TRUE
■ FALSE

a3) $\{3, 4\} \subseteq A$

☐ TRUE
■ FALSE

a4) $\varnothing \in A$

☐ TRUE
■ FALSE

a5) $\{\varnothing\} \in A$

■ TRUE
☐ FALSE

a6) $\varnothing \subseteq \mathcal{P}(A)$

■ TRUE
☐ FALSE

(b) [3 points] Explain briefly and fully your answer for one (and only one) among a1, a2, a3, a4, a5, a6.

The explanation for

☐ a1 ☐ a2 ☐ a3 ☐ a4 ☐ a5 ■ a6

is:

Solution

The empty set is a subset of every set.

(c) [13 points] List all the elements of the following sets:

c1) $A \cap \mathbb{N} = \{3\}$

c2) $A \cup \{\varnothing, 1, 2, 3, 4\} = \{\varnothing, \{\varnothing\}, 1, 2, 3, 4, \{3, 4\}\}$

c3) $A \setminus \mathbb{Q} = \{\{\varnothing\}, \{3, 4\}\}$

c4) $\mathcal{P}(A) = \{\varnothing, \{\{\varnothing\}\}, \{3\}, \{\{3, 4\}\}, \{\{\varnothing\}, 3\}, \{\{\varnothing\}, \{3, 4\}\}, \{3, \{3, 4\}\}, \{\varnothing\}, 3, \{3, 4\}\}\}$

c5) $A \cap \mathcal{P}(A) = \varnothing$

Exercise 2 (16 points)

Describe the following sets with a *set-builder notation*, i.e. as truth set of an open sentence. Remember that we use the convention that $\mathbb{N} = \{1, 2, 3, \ldots\}$.

a) $A = \{\ldots, -2, 0, 2, 4, 6, 8 \ldots\ldots\}$

Solution

$A = \{n \in \mathbb{Z} \,|\, n = 2k,\ k \in \mathbb{Z}\}$.

b) $B = \{0, 10, 20, 30, 40, \ldots, 1000\}$

Solution

$B = \{n \in \mathbb{Z} \,|\, n = 10k,\ k \in \mathbb{Z},\ 0 \leq k \leq 100\}$.

c) $C = \left\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots\right\}$

Solution

$C = \left\{x \in \mathbb{Q} \,|\, x = \frac{1}{3^k},\ k \in \mathbb{N} \cup \{0\}\right\}$.

d) $D = \left\{\ldots - \frac{5}{4}, -\frac{4}{3}, -\frac{3}{2}, -\frac{2}{1}\right\}$

Solution

$D = \left\{x \in \mathbb{Q} \,|\, x = -\frac{n+1}{n},\ n \in \mathbb{N}\right\}$.

Exercise 3 (18 points)

(a) [10 points] Prove **by contrapositive** the following claim (please, write down the contrapositive of the statement first):

Claim 1: *Let* $n \in \mathbb{Z}$. If n^3 *is even* then n *is even.*

Solution

Contrapositive: Let $n \in \mathbb{Z}$. If n is odd, then n^3 is odd.

Proof. We will prove the claim by proving its contrapositive. Let $n \in \mathbb{Z}$. Assume that n is odd. Then, by definition, there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. So we have:

$$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1,$$

which implies that n^3 is also odd. □

(b) [8 points] Prove the following claim:

Claim 2: *For all* $n \in \mathbb{Z}$, n *is even* if and only if n^3 *is even.*

You may use Claim 1 for some part of the proof. In this case, no need of proving Claim 1 again.

Solution

Proof.

$\Rightarrow$) Assume that n is an even integer. Then, by definition, there exists $k \in \mathbb{Z}$ such that $n = 2k$. So we have:

$$n^3 = (2k)^3 = 8k^3 = 2 \cdot (4k^3),$$

which implies that n^3 is also even.

$\Leftarrow$) This is Claim 1, so we proved it in part (a) of this exercise.

□

Exercise 4 (26 points)

(a) [7 points] Let A, B, C be sets. Prove that

$$C \subseteq A \cap B \Rightarrow C \subseteq A \text{ and } C \subseteq B.$$

Solution

Proof. Let A, B, C be sets. Let us assume that $C \subseteq A \cap B$. We want to show that $C \subseteq A$ and $C \subseteq B$.
So let $x \in C$. Since $C \subseteq A \cap B$, then $x \in A \cap B$, which implies that $x \in A$ and $x \in B$.
Therefore $C \subseteq A$ and $C \subseteq B$. □

(b) [7 points] Prove that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Solution

Proof. Let A and B be sets. Let $C \in P(A \cap B)$. Then, by definition of power set, $C \subseteq A \cap B$. As proved in (a), this implies that $C \subseteq A$ and $C \subseteq B$. So $C \in \mathcal{P}(A)$ and $C \in \mathcal{P}(B)$, which implies that $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
In conclusion $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$. □

(c) [6 points] Diego says:

"*If* $C \subseteq A \cup B$ *then* $C \subseteq A$ *or* $C \subseteq B$."

Do you agree with Diego? If yes prove his statement, otherwise show a counterexample.

Solution

No, I do not agree with Diego. In order to that what Diego is claiming is false, we need to provide an example of three sets A, B, C such that $C \subseteq A \cup B$ and $C \nsubseteq A$ and $C \nsubseteq B$. Consider the following sets:

- $A = \{1, 2\}$,
- $B = \{3, 4\}$,
- $C = \{2, 3\}$.

Then $C \subseteq A \cup B = \{1, 2, 3, 4\}$, but $C \nsubseteq A$ ($3 \in C$ but $3 \notin A$) and $C \nsubseteq B$ ($2 \in C$ but $2 \notin B$).

(d) [6 points] Build an example of two sets A and B such that $\mathcal{P}(A \cup B) \nsubseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Solution

Consider the sets A and B described in (c):

- $A = \{1, 2\}$,
- $B = \{3, 4\}$.

Then we have $A \cup B = \{1, 2, 3, 4\}$. In order to show that $\mathcal{P}(A \cup B) \nsubseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ we need to find an element $C \in \mathcal{P}(A \cup B)$ such that $C \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.
Consider $C := \{2, 3\} \in \mathcal{P}(A \cup B)$ (since $C \subseteq A \cup B$), but $C \notin \mathcal{P}(A)$ (since $C \nsubseteq A$) and $C \notin \mathcal{P}(B)$ (since $C \nsubseteq B$). Therefore $C \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

Exercise 5 (13 points)

Prove **by induction** that the sum of the first n natural even integers is equal to $n(n+1)$, i.e. prove that

$$\forall n \in \mathbb{N},\, 2+4+\cdots+2n = n(n+1).$$

Solution

Proof:

Let $P(n) =$ "$2+4+\cdots+2n = n(n+1)$". We will prove the claim by induction.

- Basic step: For $n = 1$, we have $P(1) =$ "$2 = 1 \cdot 2$" $=$ "$2 = 2$", so $P(1)$ is true.

- Inductive step: Let us assume that $P(n)$ is true, i.e. that

  $$1+3+\cdots+2n = n(n+1).$$

  We want to show that $P(n+1) =$ "$2+4+\cdots+2(n+1) = (n+1)(n+2)$" is also true. We have:

  $$2+3+\cdots+(2n+2) = \underbrace{2+4+\cdots+(2n)}_{=n(n+1)} + (2n+2) =$$

  $$= n(n+1) + 2n + 1 = n^2 + 3n + 2 = (n+1)(n+2).$$

  Note that we used our inductive hypothesis in the second equality. Therefore $P(n+1)$ is also true.

In conclusion, by induction, $\forall n \in \mathbb{N}$, $2+4+\cdots+(2n) = n(n+1)$.

Exercise 6 (10 points)

Prove **by contradiction** the following claim:

Claim 3: $\sqrt[3]{2}$ *is irrational.*

If at some point of your proof you make use of any result proved in class, in a homework or in a previous exercise of this test, please state it (but there is no need of proving it again).

Solution

Proof:

Assume to the contrary that $\sqrt[3]{2}$ is a rational number, i.e. that there exist integers a and b, with $b \neq 0$, such that

$$\sqrt[3]{2} = \frac{a}{b}.$$

Without loss of generality we can also assume that a and b have no common factors (otherwise we can simplify the fraction). We have:

$$\sqrt[3]{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^3}{b^3} \overset{b\neq 0}{\Rightarrow} a^3 = 2b^3.$$

So a^3 is even, which implies that a is even (see Exercise 3). Then, by definition, there exists $k \in \mathbb{Z}$ such that $a = 2k$. By substituting in $a^3 = 2b^3$ we obtain:

$$(2k)^3 = 2b^3 \Rightarrow 8k^3 = 2b^3 \Rightarrow b^3 = 4k^3 = 2 \cdot (2k^3) \Rightarrow b^3 \text{ is even } \Rightarrow b \text{ is even.}$$

In conclusion a and b are both even, so 2 is a common factor. This contradicts the assumption that a and b have no common factors. Therefore $\sqrt[3]{2}$ is an irrational number.

Exercise 7 (10 points)

Let $a, b \in \mathbb{Z}$. Prove that if $4 \mid (a^2 + b^2)$ then a is even or b is even.

Solution

Proof:

Assume to the contrary that $4 \mid (a^2 + b^2)$ and a and b are odd. Then by definition there exist $k, h, \ell \in \mathbb{Z}$ such that $a^2 + b^2 = 4k$, $a = 2h + 1$ and $b = 2\ell + 1$. So we get:

$$a^2 + b^2 = 4k$$
$$\Downarrow$$
$$(2h+1)^2 + (2\ell+1)^2 = 4k$$
$$\Downarrow$$
$$(4h^2 + 4h + 1) + (4\ell^2 + 4\ell + 1) = 4k,$$
$$\Downarrow$$
$$4h^2 + 4h + 4\ell^2 + 4\ell + 2 = 4k$$
$$\Downarrow$$
$$2(2h^2 + 2h + 2\ell^2 + 2\ell + 1) = 4k$$
$$\Downarrow$$
$$2h^2 + 2h + 2\ell^2 + 2\ell + 1 = 2k$$
$$\Downarrow$$
$$\underbrace{2(h^2 + h + \ell^2 + \ell) + 1}_{\text{odd}} = \underbrace{2k}_{\text{even}}$$

But an integer can not be even and odd at the same time, so we have a contradiction. Therefore if $4 \mid (a^2 + b^2)$ then a is even or b is even.