Calculus I - MAC 2311 - Section 007

Homework - Review Test 3 - Solutions

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Ex 1. Compute the following limits. If you use l'Hospital's Rule state which type of indeterminate form you have.

a) $\lim_{x\to\infty} \frac{\ln(1+x^2)}{x^2}$

Solution:

We have $\lim_{x\to\infty} \ln(1+x^2) = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so that we are faced with the indeterminate form $\frac{\infty}{\infty}$. Hence we can directly apply L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{\ln(1+x^2)}{x^2} = \lim_{x\to\infty} \frac{\left(\ln(1+x^2)\right)'}{(x^2)'} = \lim_{x\to\infty} \frac{\frac{2x}{1+x^2}}{2x} = \lim_{x\to\infty} \frac{2x}{2x(1+x^2)} = \lim_{x\to\infty} \frac{1}{1+x^2} = 0.$$

b) $\lim_{x\to 0} \frac{\sin(\pi e^x)}{x}$

Solution:

We have $\lim_{x\to 0} \sin(\pi e^x) = \sin(\pi) = 0$ and $\lim_{x\to 0} x = 0$, so that we are faced with the indeterminate form $\frac{0}{0}$. Hence we can directly apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\sin(\pi e^x)}{x} = \lim_{x\to 0} \frac{\left(\sin(\pi e^x)\right)'}{(x)'} = \lim_{x\to 0} \frac{\cos(\pi e^x)\cdot \pi e^x}{1} = \cos(\pi e^0)\cdot \pi e^0 = \cos(\pi)\cdot\pi = -\pi.$$

c) $\lim_{x\to\infty} \frac{e^{-x}+1}{x}$

Solution:

We recall that $\lim_{x\to\infty} e^{-x} = 0$, so that:

$$\lim_{x\to\infty} \frac{e^{-x}+1}{x} = \frac{\lim_{x\to\infty}(e^{-x}+1)}{\lim_{x\to\infty} x} = \text{“}\frac{0+1}{\infty}\text{”} = \text{“}\frac{1}{\infty}\text{”} = 0.$$

Hence, for computing this limit we do not need to use L'Hospital's rule (actually we can not apply it, since the limit does not involve any indeterminate form).

d) $\lim_{x\to 0^+} (e^x+x)^{\frac{1}{x}}$

Solution:

Let us set:

$$y = (e^x+x)^{\frac{1}{x}}.$$

We have:

$$\lim_{x\to 0^+} (e^x+x)^{\frac{1}{x}} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln(y)} = e^{\lim_{x\to 0^+}\ln(y)}.$$

Thus, all we have to do is to compute $\lim_{x\to 0^+}\ln(y)$:

$$\lim_{x\to 0^+}\ln(y) = \lim_{x\to 0^+}\ln\left((e^x+x)^{\frac{1}{x}}\right) = \lim_{x\to 0^+}\frac{1}{x}\ln(e^x+x) = \lim_{x\to 0^+}\frac{\ln(e^x+x)}{x}.$$

Now we have $\lim_{x\to 0^+}\ln(e^x+x) = \ln(e^0+0) = \ln(1) = 0$ and $\lim_{x\to 0^+} x = 0$, so that we are faced with the indeterminate form $\frac{0}{0}$. Hence we can apply L'Hospital's Rule:

$$\lim_{x\to 0^+}\frac{\ln(e^x+x)}{x} = \lim_{x\to 0^+}\frac{\frac{e^x+1}{e^x+x}}{1} = \lim_{x\to 0^+}\frac{e^x+1}{e^x+x} = \frac{e^0+1}{e^0+0} = \frac{2}{1} = 2.$$

Hence we get $\lim_{x\to 0}\ln(y) = 2$ so that

$$\lim_{x\to 0^+}(e^x+x)^{\frac{1}{x}} = e^{\lim_{x\to 0^+}\ln(y)} = e^2.$$

e) $\lim_{x\to\infty} x\left(\frac{\pi}{2} - \tan^{-1}(x)\right)$

Solution:

We have $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1}(x)\right) = \frac{\pi}{2} - \lim_{x\to\infty}\tan^{-1}(x) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, so that we are faced with the indeterminate form $\infty\cdot 0$. Hence we rewrite the limit in the following way:

$$\lim_{x\to\infty}\frac{\frac{\pi}{2} - \tan^{-1}(x)}{\frac{1}{x}}$$

Now the indeterminate form is $\frac{0}{0}$ and we can apply L'Hospital's Rule:

$$\lim_{x\to\infty}\frac{\frac{\pi}{2} - \tan^{-1}(x)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\left(\frac{\pi}{2} - \tan^{-1}(x)\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty}\frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{x^2}{1+x^2} = 1.$$

Ex 2. After their romantic dinner at the intersection of Bruce B. Downs and Fowler Avenue, the alligators from HW 2 decide to hold hands and take a walk along Fowler Avenue. Their position after t hours was

$$f(t) = \frac{\pi}{4} - \arctan((t-1)^2) \quad \text{miles.}$$

Which is the farthest point from the intersection reached by the alligators between 0 and 2 hours?

Solution:

The problem boils down into a problem of **finding the absolute maximum and minimum values of the continuous function** $f(t) = \frac{\pi}{4} - \arctan((t-1)^2)$ **on the closed interval** $[0,2]$. Their existence is guaranteed by the Extreme Value Theorem.

- *Find the critical numbers of f and their corresponding values.*

  The function f is continuous and differentiable on $\mathbb{R}$, thus in particular on $[0,2]$. Hence, its critical numbers are all the numbers c such that $f'(c) = 0$.

Here we have:

$$f'(x) = -\frac{1}{1+((t-1)^2)^2} \cdot ((t-1)^2)' = -\frac{2(t-1)}{1+((t-1)^2)^2}.$$

Thus $f'(x) = 0$ if and only if $2(t-1) = 0$, i.e. if and only if $t = 1$. The corresponding value at $t = 1$ is $f(1) = \frac{\pi}{4} - \arctan(0) = \frac{\pi}{4}$.

- *Find the values of f at the endpoints of the interval* $[0, 2]$.

  We have $f(0) = \frac{\pi}{4} - \arctan(1) = \frac{\pi}{4} - \frac{\pi}{4} = 0$ and $f(2) = \frac{\pi}{4} - \arctan(1) = \frac{\pi}{4} - \frac{\pi}{4} = 0$.

- *Compare the values obtained in step 1 and step 2 and return the absolute maximum and the absolute minimum values of f.*

  The absolute maximum value of f on $[0, 2]$ is given by $\frac{\pi}{4}$ miles and the absolute minimum value is given by 0 miles.

We conclude that the farthest point from the intersection reached by the alligators between 0 and 2 hours is distant $\frac{\pi}{4}$ miles (*remark*: this is not the case, but if the absolute minimum value was in absolute value greater than $\frac{\pi}{4}$, this would have been the farthest point from the intersection. Indeed we are considering the distance - i.e. the absolute value - of the obtained points from the intersection).

Ex 3. Consider the function

$$f(x) = \frac{1}{x} + x + 1.$$

a) Find the domain of definition of f.

b) Find the horizontal and vertical asymptotes.

c) Find the critical numbers of f.

d) Find the intervals of increase/decrease of f and the local maxima/minima of f.

e) Find the intervals where f concaves upward/downward and the inflection points of f.

f) Sketch the graph of $y = f(x)$, by using the information you collected above.

Solution:

a) *Find the domain of definition of f.*

   $D = \mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$.

b) *Find the horizontal and vertical asymptotes.*

For finding the possible horizontal and vertical asymptotes we have to study the behavior of the function at the endpoints of the domain, which are in this case $-\infty, 0^-, 0^+, \infty$.

- ⋆ *Horizontal asymptotes*

  $\lim_{x\to\infty} \frac{1}{x} + x + 1 =$ "$0 + \infty + 1$" $= \infty$.
  $\lim_{x\to-\infty} \frac{1}{x} + x + 1 =$ "$0 - \infty + 1$" $= -\infty$.

  Since these two limits are not finite, the function has no horizontal asymptotes.

- ⋆ *Vertical asymptotes*

  $\lim_{x\to 0^-} \frac{1}{x} + x + 1 =$ "$-\infty + 0 + 1$" $= -\infty$.
  $\lim_{x\to 0^+} \frac{1}{x} + x + 1 =$ "$\infty + 0 + 1$" $= \infty$.

  Hence 0 is an infinite discontinuity and $x = 0$ is the corresponding vertical asymptote.

c) *Find the critical numbers of f.*

The critical numbers of f are the numbers c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist. Let us compute $f'(x)$.

$$f'(x) = \left(\frac{1}{x} + x + 1\right)' = -\frac{1}{x^2} + 1 = \frac{-1 + x^2}{x^2} = \frac{x^2 - 1}{x^2}.$$

- ⋆ $f'(c) = 0$:

  We have that $f'(x) = 0 \Leftrightarrow \frac{x^2-1}{x^2} = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow (x-1)(x+1) = 0 \Leftrightarrow$ $x = -1$ or $x = 1$, which are both in the domain D.

- ⋆ $f'(c)$ does not exist:

  The derivative f' is not defined at $x = 0$, but this point is not in the domain D.

  Hence the critical numbers are $x = -1$ and $x = 1$.

d) *Find the intervals of increase/decrease of f and the local maxima/minima of f.*

We have to study the sign of the first derivative $f'(x)$. Indeed the function is increasing in the intervals where $f'(x) > 0$ and decreasing in the intervals where $f'(x) < 0$.

		-1		0		1	
$f'(x)$	$+$		$-$		$-$		$+$
$f(x)$	↗		↘		↘		↗

Remark: On the real line we mark all the values that make the numerator or the denominator of f' equal to 0. In this case the numerator is $x^2 - 1$ and the denominator x^2, so that we consider $-1, 0$ and 1. Now, in order to determine the sign of $f'(x)$ on the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, -\infty)$, we simply plug in into $f'(x)$ a number inside the previous intervals and we keep the sign of the obtained value. For example $-2 \in (-\infty, -1)$ and $f'(-2) = \frac{3}{4} > 0$, so that $f'(x) > 0$ on $(-\infty, -1)$.

We conclude that the function f is increasing on the interval $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 0) \cup (0, 1)$. We obtain also that $x = -1$ is a local maximum point (which corresponds to the local maximum value $f(-1) = 1$) and $x = 1$ is a local minimum point (which corresponds to the local minimum value $f(1) = 3$).

e) *Find the intervals where f concaves upward/downward and the inflection points of f.*

We have to study the sign of the second derivative $f''(x)$. Indeed the function concaves upward in the intervals where $f''(x) > 0$ and concaves downward in the intervals where $f''(x) < 0$.
The inflection points are the points where f is continuous and the graph of f switches from being upward to downward, or vice versa.
Let us first compute $f''(x)$:

$$f''(x) = \left(\frac{x^2 - 1}{x^2}\right)' = \left(1 - \frac{1}{x^2}\right)' = \left(1 - x^{-2}\right)' = -(-2x^{-3}) = \frac{2}{x^3}.$$

		0	
$f''(x)$	$-$		$+$
$f(x)$	DOWN		UP

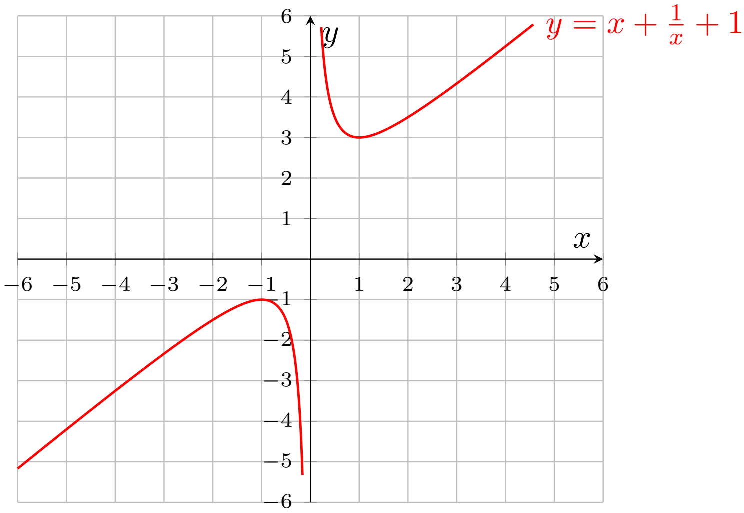
We conclude that the function f concaves downward on the interval $(-\infty, 0)$ and upward on $(0, \infty)$.

Attention: Even if at $x = 0$ the graph of the function switches from being concave downward to concave upward, this does not correspond to an inflection point, since f is not continuous at $x = 0$ (actually 0 does not belong to the domain of f).

f) *Sketch the graph of $y = f(x)$, by using the information you collected above.*

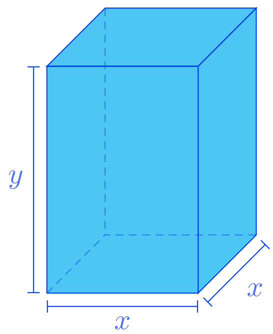
In the previous steps we obtained the following information:

- ⋆ $D = \mathbb{R} \backslash \{0\} = (-\infty, 0) \cup (0, \infty)$.
- ⋆ There are no horizontal asymptotes and $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.
- ⋆ The line $x = 0$ is a vertical asymptote and $\lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0^+} f(x) = \infty$.
- ⋆ The function f is increasing on the interval $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 0) \cup (0, 1)$. Moreover $x = -1$ is a local maximum point (which corresponds to the local maximum value $f(-1) = 1$) and $x = 1$ is a local minimum point (which corresponds to the local minimum value $f(1) = 3$). Then the graph of f passes through the points $(-1, 1)$ and $(1, 3)$.
- ⋆ The function f concaves downward on the interval $(-\infty, 0)$ and upward on $(0, \infty)$ and there are no inflection points.



Ex 4. Among all boxes with a square base and volume 27 cm^3, what are the dimensions of the box which minimize the surface area?

Solution: Let us consider a box with square base and volume 27 cm^3.



Let us call:

x: the side length of the base of the box (the sides are all of same length since the base is a square).
y: the height of the box.

Since the box has prescribed volume 27 cm^3, the variables x and y satisfy the following **constraint equation:**

$$\textbf{Volume} : x^2y = 27.$$

We want to **minimize** the function of the surface area. The surface area of a box is given by the sum of the areas of the 6 rectangles that cover its surface. In this case we have:

$$\textbf{Surface area} : 2x^2 + 4xy.$$

Now, all we have to do is obtaining from this function, which is a priori a function in two variables, a function in only one variable (indifferently in x or y), and applying the classical tools for finding the local minimum point/value.

From the constraint equation we get

$$y = \frac{27}{x^2}.$$

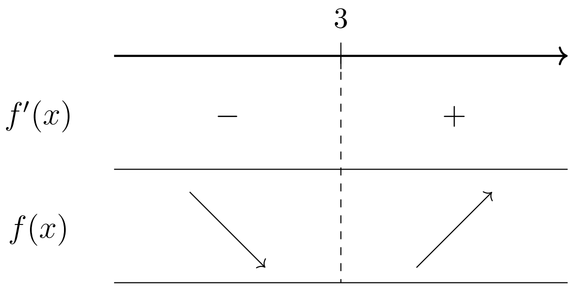
If we replace this in the surface area function $2x^2 + 4xy$ we obtain the following function in one variable:

$$f(x) = 2x^2 + 4x\frac{27}{x^2} = 2x^2 + \frac{4 \cdot 27}{x}.$$

Thus, let us find the critical points of $f(x)$:

$$f'(x) = 4x - \frac{4 \cdot 27}{x^2} = \frac{4x^3 - 4 \cdot 27}{x^2} = \frac{4(x^3 - 27)}{x^2} = 0 \Leftrightarrow 4(x^3 - 27) = 0 \Leftrightarrow x^3 = 27 \Leftrightarrow x = 3.$$

Moreover we have:

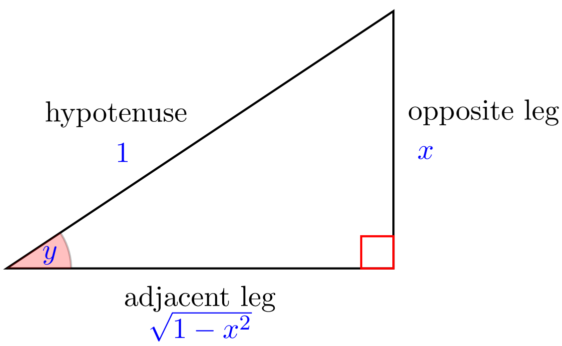


Thus $x = 3$ is a local minimum point (and also the absolute minimum point of f). Hence we obtain that the dimensions of the box of volume 27 cm^3 which minimize the surface area are $x = 3$ cm and $y = \frac{27}{x^2} = 3$ cm. The box is actually a **cube**.

Ex 5. Which statements are True/False? Justify your answers.

a) We have $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$ for all x in $[-1, 1]$.

True. Let us set $y = \sin^{-1}(x)$. Then $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. We recall that in a right triangle $\sin(y) = \frac{\text{opposite leg}}{\text{hypotenuse}}$. Here $\sin(y) = \frac{x}{1}$, hence we can consider the right triangle with hypotenuse of length 1 and opposite leg of length x (see the picture below):



Then:

$$\cos(\sin^{-1}(2x)) = \cos(y) = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}, \text{ for all } x \text{ in } [-1, 1].$$

b) If f is a function which is continuous on $[a, b]$, differentiable on (a, b) and such that $f(a) = f(b)$ then f has at least one critical point in (a, b).

True. If f is a function which is continuous on $[a, b]$, differentiable on (a, b) and such that $f(a) = f(b)$, then by Rolle's theorem there exists a number c in (a, b) such that $f'(c) = 0$. This number c is, by definition, a critical point for f and it is in (a, b).

c) There exists a function f such that $f(0) = 0$, $f(8) = 8$ and $f'(x) \geq 16$ for all x in $[0, 8]$.

False. Since $f'(x)$ is defined for all x in $[0, 8]$, the function f is differentiable (then continuous) on $[0, 8]$. By the Mean Value Theorem there exists a number c in $(0, 8)$ such that

$$f'(c) = \frac{f(8) - f(0)}{8 - 0} = \frac{8 - 0}{8} = 1.$$

Then it is not true that $f'(x) \geq 16$ for all x in $[0, 8]$.

d) If $f'(x) = g'(x)$ for all x in $\mathbb{R}$, then $f(x) = g(x)$.

False. If $f'(x) = g'(x)$ for all x in $\mathbb{R}$ then $(f - g)'(x) = 0$ for all x in $\mathbb{R}$. This implies that $f - g = c$ where c is a constant (possibly different from 0), i.e. $f = g + c$. For example the functions $f = x$ and $g = x + 1$ have same derivative $f'(x) = g'(x) = 1$ but are different.