Calculus I - MAC 2311 - Section 007

Quiz 5 - Solutions

10/26/2017

1) [3 points] Without using a calculator compute the following values.

a) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

b) $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

c) $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

d) $\sin\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

e) $\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

f) $\cos^{-1}\left(\sqrt{3}\sin\left(\frac{\pi}{6}\right)\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

2) [1+2.5+1.5 points] Consider the function

$$f(x) = \tan(\sin^{-1}(2x)).$$

a) Find the domain of f.

The function f is a composition of the functions tan and $\sin^{-1}$ which have respectively domain $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, \text{ where } k \text{ is an integer }\}$ and $[-1, 1]$. Hence x is in the domain D of f if and only if:

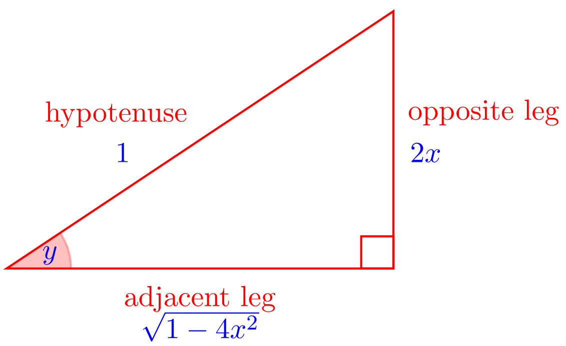
$$\begin{cases} -\frac{\pi}{2} < \sin^{-1}(2x) < \frac{\pi}{2} \\ -1 \leq 2x \leq 1 \end{cases} \Leftrightarrow \begin{cases} -1 < 2x < 1 \\ -\frac{1}{2} \leq x \leq \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} -\frac{1}{2} < x < \frac{1}{2} \\ -\frac{1}{2} \leq x \leq \frac{1}{2} \end{cases} \Leftrightarrow -\frac{1}{2} < x < \frac{1}{2}$$

We obtain that $D = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

b) Show that for each x in the domain a simplified expression for f is $\frac{2x}{\sqrt{1-4x^2}}$.

Let us set $y = \sin^{-1}(2x)$. Then $\sin(y) = 2x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (since $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$).

We recall that in a right triangle $\sin(y) = \frac{\text{opposite leg}}{\text{hypotenuse}}$. Here $\sin(y) = \frac{2x}{1}$, hence we can consider the right triangle with hypotenuse of length 1 and opposite leg of length $2x$ (see the picture below):



Hence:

$$\tan(\sin^{-1}(2x)) = \tan(y) = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{2x}{\sqrt{1-4x^2}}$$

c) Compute $f'(x)$.

Since we have that $f(x) = \tan(\sin^{-1}(2x)) = \dfrac{2x}{\sqrt{1-4x^2}}$ for all x in D, we can compute the derivative of f in two different ways.

<u>I method</u>: $f(x) = \tan(\sin^{-1}(2x))$. Then

$$\begin{aligned} f'(x) &= \sec^2(\sin^{-1}(2x)) \cdot (\sin^{-1}(2x))' = \\ &= \sec^2(\sin^{-1}(2x)) \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot (2x)' = \\ &= \sec^2(\sin^{-1}(2x)) \cdot \frac{2}{\sqrt{1-4x^2}} \end{aligned}$$

<u>II method</u>: $f(x) = \dfrac{2x}{\sqrt{1-4x^2}}$. Then

$$\begin{aligned} f'(x) &= \frac{(2x)' \cdot \sqrt{1-4x^2} - 2x(\sqrt{1-4x^2})'}{1-4x^2} = \\ &= \frac{(2x)' \cdot \sqrt{1-4x^2} - 2x((1-4x^2)^{\frac{1}{2}})'}{1-4x^2} = \\ &= \frac{2\sqrt{1-4x^2} - 2x\left(\frac{1}{2}(1-4x^2)^{-\frac{1}{2}}(-8x)\right)}{1-4x^2} \\ &= \frac{2\sqrt{1-4x^2} + 8x^2(1-4x^2)^{-\frac{1}{2}}}{1-4x^2} \end{aligned}$$

3) [1.5+1.5 points] Compute the following limits and show all your work:

a) $\displaystyle\lim_{x\to 0} \frac{x^2}{1-\cos(x)}$

We have $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} 1-\cos(x) = 0$, so that we are faced with the indeterminate form $\frac{0}{0}$. Hence we can apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{x^2}{1-\cos(x)} = \lim_{x\to 0} \frac{(x^2)'}{(1-\cos(x))'} = \lim_{x\to 0} \frac{2x}{\sin(x)}.$$

Now we can proceed in two different ways.

<u>I method</u>: We can use the special limit $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1$:

$$\lim_{x\to 0} \frac{2x}{\sin(x)} = \lim_{x\to 0} \frac{2x}{\sin(x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0} \frac{2}{\frac{\sin(x)}{x}} = \frac{2}{1} = 2.$$

<u>II method</u>: We can apply again L'Hospital's Rule:

$$\lim_{x\to 0}\frac{2x}{\sin(x)} = \lim_{x\to 0}\frac{(2x)'}{(\sin(x))'} = \lim_{x\to 0}\frac{2}{\cos(x)} = \frac{2}{1} = 2.$$

Therefore

$$\lim_{x\to 0}\frac{x^2}{1-\cos(x)} = \mathbf{2}.$$

b) $\lim\limits_{x\to\infty} xe^{-x}$

We have $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} e^{-x} = 0$, so that we are faced with the indeterminate form $\infty \cdot 0$. Hence we rewrite the limit in the following way

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x}$$

Now the indeterminate form is $\frac{\infty}{\infty}$ and we can apply L'Hospital's Rule:

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{(x)'}{(e^x)'} = \lim_{x\to\infty}\frac{1}{e^x} = 0.$$

Therefore

$$\lim_{x\to\infty} xe^{-x} = \mathbf{0}.$$