Calculating limits

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In the following tables the writing "$\lim_{x\to\square} f(x)$" stands for $\lim_{x\to a} f(x)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, L_1 and L_2 are real numbers (possibly equal to 0, unless otherwise specified) and the symbol ☠ means an *indeterminate form* (we recall that a form of limit is said to be *indeterminate* when knowing the limiting behavior of individual parts of the expression is not sufficient to actually determine the overall limit).

Sum

$\lim_{x\to\square} f(x)$	$\lim_{x\to\square} g(x)$	$\lim_{x\to\square} f(x)+g(x)$
L_1	L_2	L_1+L_2
L_1	∞	∞
L_1	$-\infty$	$-\infty$
∞	L_2	∞
∞	∞	∞
∞	$-\infty$	☠
$-\infty$	L_2	$-\infty$
$-\infty$	∞	☠
$-\infty$	$-\infty$	$-\infty$

We can consider the limit of the difference of two functions as the limit of a sum in the following way:

$$\lim_{x\to\square} f(x)-g(x) = \lim_{x\to\square} f(x)+(-g(x)).$$

Hence, for example, if $\lim_{x\to\square} f(x)=\infty$ and $\lim_{x\to\square} g(x)=-\infty$ we have $\lim_{x\to\square} f(x)-g(x) =$ "$\infty-(-\infty)$" $=$ "$\infty+\infty$" $=\infty$.

Examples.

1) $\lim_{x\to-\infty} \sqrt{3-4x}-x+1 = \lim_{x\to-\infty}(\sqrt{3-4x}) + \lim_{x\to-\infty}(-x) + \lim_{x\to-\infty} 1 =$ "$\infty+\infty-1$" $=\infty$.

2) $\lim_{x\to\infty} x^2-x = \lim_{x\to\infty}(x^2) + \lim_{x\to\infty}(-x) =$ "$\infty-\infty$" $\to$ ☠ (look at the examples of the product for seeing how to escape to the indeterminate form...)

PRODUCT

$\lim_{x\to\square} f(x)$	$\lim_{x\to\square} g(x)$	$\lim_{x\to\square} f(x)g(x)$
L_1	L_2	$L_1 \cdot L_2$
$L_1 > 0$	∞	∞
$L_1 > 0$	$-\infty$	$-\infty$
0	∞	☠
0	$-\infty$	☠
$L_1 < 0$	∞	$-\infty$
$L_1 < 0$	$-\infty$	∞
∞	∞	∞
∞	$-\infty$	$-\infty$

The table for the product can be completed by using the commutative property of the product (that is the reason why in the table for example the case $\lim_{x\to\square} f(x) = \infty$ and $\lim_{x\to\square} g(x) = L_2$ does not appear).

Moreover we can deduce the table for the limit of the quotient of two functions by considering the quotient as a product:

$$\lim_{x\to\square} \frac{f(x)}{g(x)} = \lim_{x\to\square} f(x) \cdot \frac{1}{g(x)}$$

and using the following table:

$\lim_{x\to\square} g(x)$	$\lim_{x\to\square} \frac{1}{g(x)}$
L	$\frac{1}{L}$
$0^+\ (>0)$	∞
$0^-\ (<0)$	$-\infty$
∞	0

We deduce that also $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ are indeterminate forms ☠.

Examples.

1) $\lim_{x\to\infty} x^2 - x = \lim_{x\to\infty} x(x-1) = \lim_{x\to\infty} x \cdot \lim_{x\to\infty} (x-1) = \text{“}\infty \cdot \infty\text{”} = \infty$

2) $\lim_{x\to\frac{\pi}{2}^+} \frac{1}{\cos x} + \frac{1}{\frac{\pi}{2} - x} = \text{“}\frac{1}{0^-} + \frac{1}{0^-}\text{”} = \text{“} - \infty - \infty\text{”} = -\infty.$

The case of rational functions

We recall that a rational function is a function of the form:

$$f(x)=\frac{P(x)}{Q(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}$$

where $P(x)$ and $Q(x)$ are two polynomials with real coefficients of degree n and m respectively ($a_n\neq 0, b_m\neq 0$).

We consider here the particular limits

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)} \qquad \text{or} \qquad \lim_{x\to-\infty}\frac{P(x)}{Q(x)}.$$

Theorem 1. *We have:*

$$\lim_{x\to\pm\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}=\lim_{x\to\pm\infty}\frac{a_nx^n}{b_mx^m}$$

Proof.

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}&=\lim_{x\to\infty}\frac{x^n(a_n+a_{n-1}\frac{1}{x}+\cdots+a_0\frac{1}{x^n})}{x^m(b_m+b_{m-1}\frac{1}{x^{m-1}}+\cdots+b_0\frac{1}{x^m})}=\\&=\lim_{x\to\pm\infty}\frac{x^n}{x^m}\cdot\lim_{x\to\pm\infty}\frac{a_n+a_{n-1}\frac{1}{x}+\cdots+a_0\frac{1}{x^n}}{b_m+b_{m-1}\frac{1}{x^{m-1}}+\cdots+b_0\frac{1}{x^m}}=\\&=\left(\lim_{x\to\pm\infty}\frac{x^n}{x^m}\right)\cdot\frac{a_n+0+\cdots+0}{b_m+0+\cdots+0}=\\&=\lim_{x\to\pm\infty}\frac{a_nx^n}{b_mx^m}.\end{aligned}$$

□

Hence the limit takes different values according to different cases:

1) $n>m$

$$\lim_{x\to\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}=\frac{a_n}{b_m}\lim_{x\to\infty}x^{n-m}=\begin{cases}\infty, & \text{if } \frac{a_n}{b_m}>0\\-\infty, & \text{if } \frac{a_n}{b_m}<0\end{cases}.$$

$$\lim_{x\to-\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}=\lim_{x\to-\infty}\frac{a_nx^n}{b_mx^m}=\frac{a_n}{b_m}\lim_{x\to-\infty}x^{n-m}=\begin{cases}\infty, & \text{if } \frac{a_n}{b_m}>0 \text{ and } n-m \text{ even}\\-\infty, & \text{if } \frac{a_n}{b_m}>0 \text{ and } n-m \text{ odd}\\-\infty, & \text{if } \frac{a_n}{b_m}<0 \text{ and } n-m \text{ even}\\\infty, & \text{if } \frac{a_n}{b_m}<0 \text{ and } n-m \text{ odd}\end{cases}.$$

2) $n=m$

$$\lim_{x\to\pm\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_nx^n+b_{n-1}x^{n-1}+\cdots+b_0}=\lim_{x\to\pm\infty}\frac{a_nx^n}{b_nx^n}=\frac{a_n}{b_n}.$$

3) $n<m$

$$\lim_{x\to\pm\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}=\lim_{x\to\pm\infty}\frac{a_nx^n}{b_mx^m}=\frac{a_n}{b_m}\lim_{x\to\pm\infty}\frac{1}{x^{m-n}}=0.$$

Examples.

1) $$\lim_{x\to\infty}\frac{3x^2-x+5}{4x^2-1}=\lim_{x\to\infty}\frac{x^2(3-\frac{1}{x}+\frac{5}{x^2})}{x^2(4-\frac{1}{x^2})}=\lim_{x\to\infty}\frac{3-\frac{1}{x}+\frac{5}{x^2}}{4-\frac{1}{x^2}}=\frac{3}{4}$$

2) $$\lim_{x\to-\infty}\frac{3x^4-2x^2+1}{-2x^2-2}=\lim_{x\to-\infty}\frac{x^4(3-2\frac{1}{x^2}+\frac{1}{x^4})}{x^2(-2-\frac{2}{x^2})}=\lim_{x\to-\infty}\frac{x^2(3-2\frac{1}{x^2}+\frac{1}{x^4})}{-2-\frac{2}{x^2}}=\text{“}\frac{\infty\cdot 3}{-2}\text{”}=-\infty$$