LIMIT OF A FUNCTION (Sec. 1.3 of the book)

The concept of limit of a function is extremely important, since all calculus is based upon it.

Indeed, we will see that the idea of limit, other than being important in itself, is also the basic notions of differential and integral calculus: the derivative and the integral of a function

LIMIT

$\lim_{x \to a} f(x)$

DERIVATIVE

$\frac{d}{dx} f(x)$

INTEGRAL

$\int f(x)\,dx$

The limit of a function concerns the behavior of that function near a particular input.

In some sense it is the prediction of the value of a function we should get at a point.

<u>ex</u>: Let us go back to a previous example:

$$f(x) = \frac{x^2+2x}{x} = \frac{x(x+2)}{x}$$

For all real numbers we can compute the value of the function, except 0. Indeed:

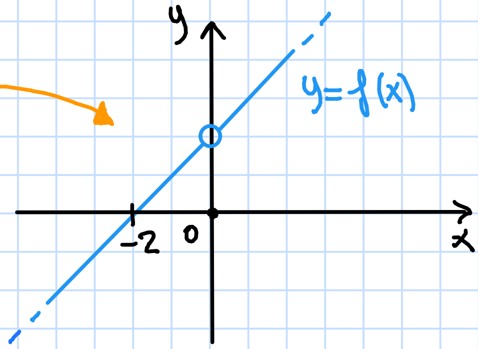
$f(0) = \text{"}\frac{0}{0}\text{"}$ which is undefined (indeed 0 does not belong to the domain of f)

But, what is the "behavior" of f "near" 0, i.e. when an input is really close to 0 (for instance -0.001, -0.00001, 0.000000001, etc...)?

THAT'S WHAT LIMITS ARE FOR!

We saw that $f(x) = x+2$ for all $x \neq 0$ and has the following graph.

Since at 0 the function is not defined we want to study what happens around zero



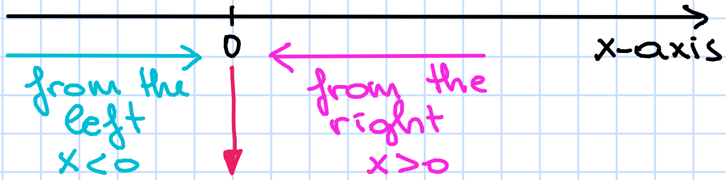
We introduce the following notation:

$$\lim_{x \to 0} f(x) = ?$$ and we read: The limit of f(x) as x approaches 0 equals...

this means that we are approaching 0 on the x-axis:
- from the left

and
- from the right

here we are looking for a y-value (which is a real number!)



but we do not care about what it is going on at x=0

When we write:

$$\lim_{x \to 0} f(x)$$

we have automatically to think that we have to check what happens both sides of 0, on the left-hand and the right-hand side.

The most intuitive way to compute this limit is to build a table of values, actually two tables of values (one for the left-hand side and one for the right-hand side).

In each table the numbers in the left column are inputs that approach more and more 0.

from the left
$x < 0$

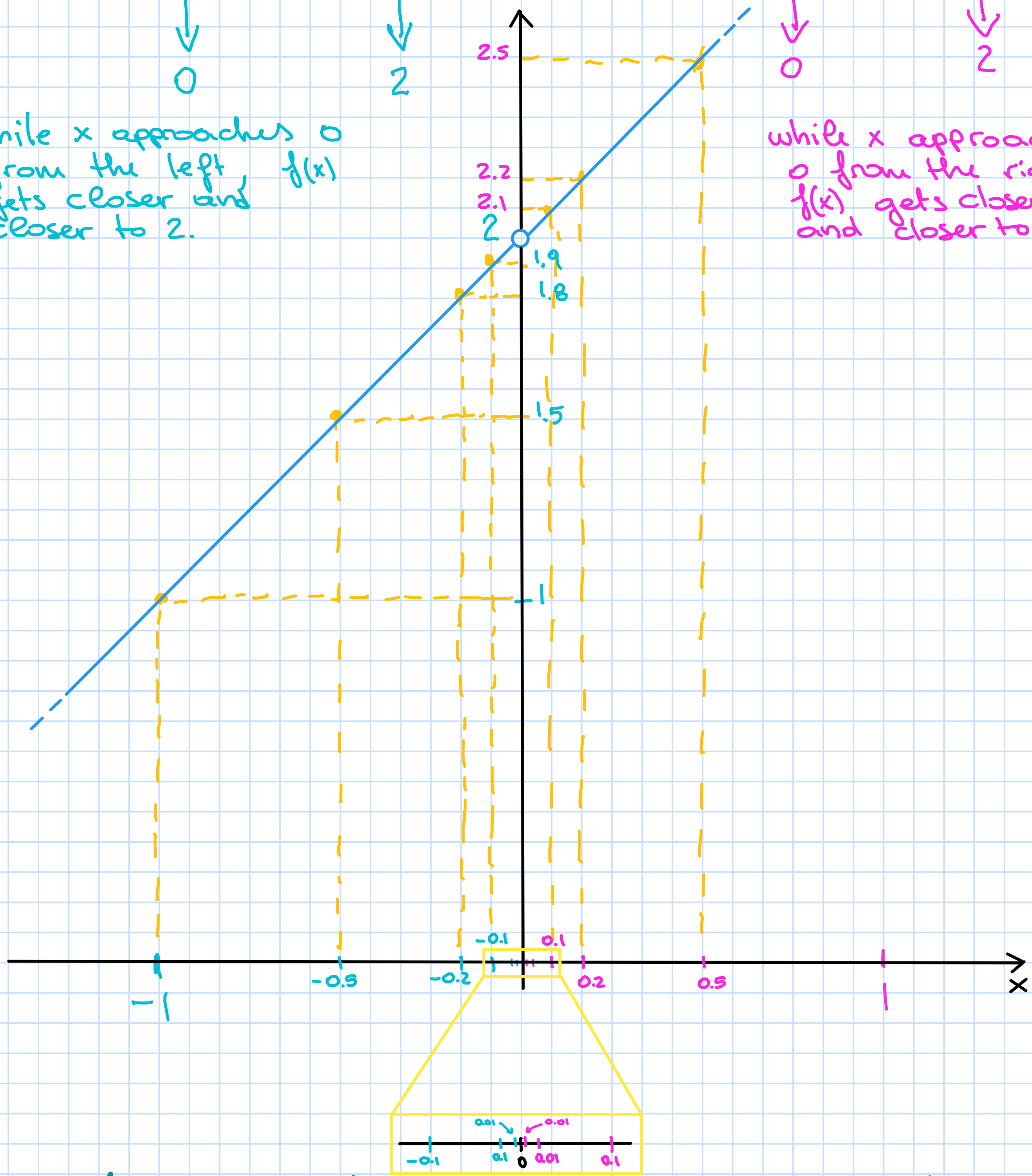
x	$f(x)$
-1	1
-0.5	1.5
-0.2	1.8
-0.1	1.9
-0.01	1.99
-0.001	1.999
-0.000001	1.999999
$\downarrow$	$\downarrow$
0	2

from the right
$x > 0$

x	$f(x)$
1	3
0.5	2.5
0.2	2.2
0.1	2.1
0.01	2.01
0.001	2.001
0.000001	2.000001
$\downarrow$	$\downarrow$
0	2

While x approaches 0 from the left $f(x)$ gets closer and closer to 2.

while x approaches 0 from the right $f(x)$ gets closer and closer to 2



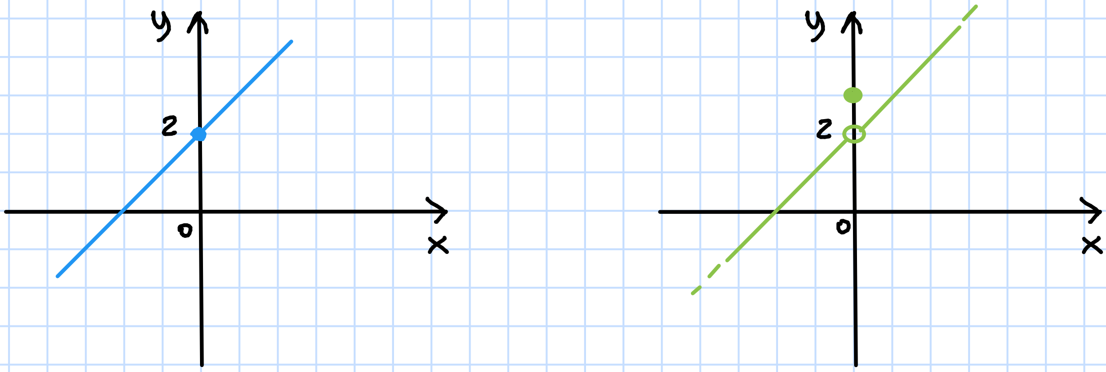
Since, from both sides, when x approaches 0 then $f(x)$ gets closer and closer to 2, we write

$$\lim_{x \to 0} f(x) = 2.$$

Question: And what about if at 0 the function was defined to be 2 or another value, that is if we are in one of the following two situations?

$g(x) = x+2$

$$h(x) = \begin{cases} \frac{x(x+2)}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$$



This does not change anything!

We have also $\lim_{x \to 0} g(x) = 2$ and $\lim_{x \to 0} h(x) = 2$.

Indeed, when we compute limits, we do not care about what the function is doing exactly at 2.

We only care about what is happening just around!

We have the following definition:

for a more formal and precise definition look inside the book (ε, δ definition)

Def: Let f be a function and a a real number. Suppose that f is defined in a neighbourhood of a (this means in some open interval that contains a except possibly at a itself).

We write

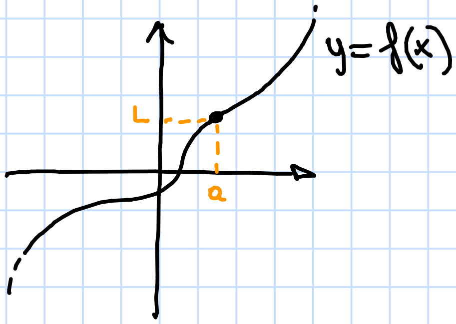
$$\lim_{x \to a} f(x) = L$$

: "the limit of $f(x)$ as x approaches a equals L"

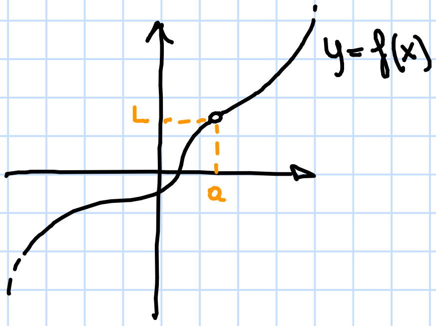
if we can make the values of $f(x)$ arbitrary closed to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Again, that means that in finding the limit we never consider $x = a$

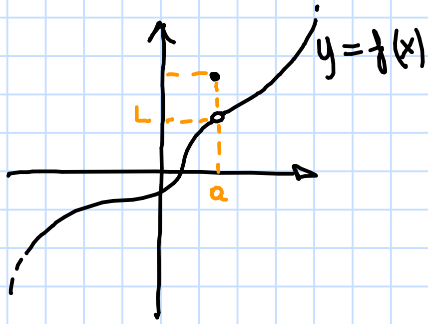
So, in either of the following cases:



$f(a) = L$



a not in the domain of f



$f(a) \neq L$

We have

$$\lim_{x \to a} f(x) = L$$

Sometimes left-hand and right-hand limits are not the same.

This leads us to consider one-sided limits.

We call and denote them:

→ left-hand limit : $\lim_{x \to a^-} f(x)$

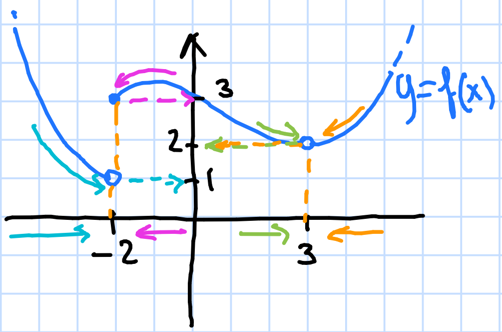
this minus here means that we are considering inputs less than a : $x < a$

→ right-hand limit : $\lim_{x \to a^+} f(x)$

this plus here means that we are considering inputs greater than a : $x > a$

Let us consider the following example:

ex:



$\lim_{x \to -2^-} f(x) = 1$

$\lim_{x \to 3^-} f(x) = 2$

$\lim_{x \to -2^+} f(x) = 3$

$\lim_{x \to 3^+} f(x) = 2$

Now:

different left-hand and right-hand limit

same left-hand and right-hand limit

$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to 2^+} f(x) \qquad\qquad \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 2$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$\lim_{x\to -2} f(x)$ does not exist (we also write DNE)

$$\lim_{x\to 3} f(x) = 2$$

Indeed the overall limit exists if and only if the left-hand and the right hand limit exist and are equal.

More precisely:

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Let us consider another example:

$$f(x) = \frac{1}{x^2+4x+4} = \frac{1}{(x+2)^2}$$

The domain of f is $D = (-\infty,-2)\cup(-2,\infty) = \mathbb{R}\setminus\{-2\}$

Hence, since $x=-2$ does not belong to D, it is interesting to compute,

$$\lim_{x\to 0} f(x)$$

Let us build again the two tables of values:

	x	$f(x)$
	-3	1
	-2.5	4
	-2.2	25
	-2.1	100
	-2.05	400
	-2.01	10000
-2	-2.001	$1{,}000{,}000$

	x	$f(x)$
	-1	1
	-1.5	4
	-1.8	25
	-1.9	100
	-1.95	400
	-1.99	$10{,}000$
-2	-1.999	$1{,}000{,}00$

We remark that when x approaches -2 $f(x)$ becomes arbitraly large.

So $\lim_{x\to 0} \frac{1}{(x+2)^2}$ does not exist because $f(x)$ does not get closer to any specific real number but it does not exist in a particular way.

We will come back to thi example in Section 1.6 !