IMPLICIT DIFFERENTIATION (Sec. 2.6)

Implicit differentiation is an application of chain rule. We need first to understand what means for a function to be implicitely defined.

Let us start from the definition of "explicit" and "implicit" that we can find in a dictionary.

EXPLICIT: stated clearly and in detail, leaving no room for confusion or doubt.

IMPLICIT: suggested but not communicated directly.

These two definitions give a very good idea of what means for a function to be explicitely or implicitely defined.

So far we have only met with functions explicitely defined. Indeed, when we write:

left side depends only on the dependent variable → $y = f(x)$ → right side depends only on the independent variable

we are defining explicitely our function. This means that the dependent variable is expressed explicitely in terms of the independent variable.

In this case we can easily compute the derivative $y' = f'(x)$ (in Lagrange notation) or $\frac{dy}{dx} = \frac{df}{dx}$ (in Leibniz notation) by applying the differentiation rules.

<u>ex</u>: $y = 3x^2 + \cos(x)$ is explicitely defined

$$\left.\begin{array}{l}\frac{dy}{dx}\\ y'\end{array}\right\} = 6x - \sin(x)$$

Now it is also possible to define a function implicitely as a function of x, through an equation that relates the indepen-dent and dependent variable.

Let us understand this on an example. Let us consider the equation:

$$x^2 + y^2 = 1$$

We will study this equation from two different points of view: algebraically and geometrically.

Algebraically

If we choose x as the independent variable, we say that y is implicitely defined as a function of x by the equation:

$$x^2 + y^2 = 1.$$

But attention: when y is implicitely defined, y is not a function in general.
Indeed if we fix an input for x there might exist several values of y that satisfy the equation, in other words several outputs!

ex: if $x=0 \Rightarrow 0^2 + y^2 = 1 \Rightarrow y^2 = 1 \Rightarrow y = 1$ or $y = -1$

if $x = \frac{1}{2} \Rightarrow \left(\frac{1}{2}\right)^2 + y^2 = 1 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \frac{\sqrt{3}}{2}$ or $y = -\frac{\sqrt{3}}{2}$

There are indeed two functions that are implicitely represented by the equation $x^2 + y^2 = 1$:

$$f_1(x) = \sqrt{1 - x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1 + x^2}$$

Note: we say that $f(x)$ is a function implicitely defined by an equation $F(x,y) = 0$ if

$$F(x, f(x)) = 0$$

for all values x in the domain of f.

Indeed for f_1 and f_2 the equalities $x^2 + (f_1(x))^2 = 1$ and $x^2 + (f_2(x))^2 = 1$ are true for all values x in $[-1,1]$, which is the domain for both f_1 and f_2.

The functions f_1 and f_2 can be found by solving the equation $x^2 + y^2 = 1$ for y.

$$x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2 \Rightarrow y = \pm\sqrt{1 - x^2}.$$

Remark: Note that each explicit function can be written implicitely:

ex: $y = 3x^2 + \cos(x) \longrightarrow \underbrace{y - 3x^2 - \cos(x) = 0}_{F(x,y)}$

Geometrically

An equation $F(x,y) = 0$ in the variables x and y corresponds in the real plane to a curve which is not in general the graph of a function.

The points of the curve are all the points of the plane whose coordinates (x,y) satisfy the equation $F(x,y) = 0$.

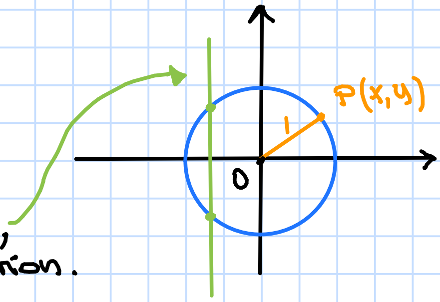
There exist also equations that correspond to curves without points

ex: $x^2 + y^2 = -1$ ← indeed the sum of two squares is always a non negative number

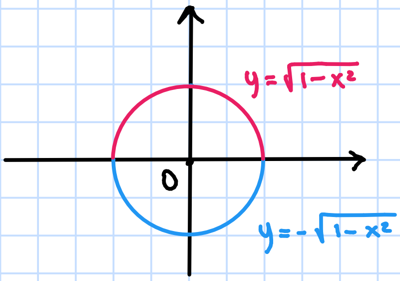
If we go back to our equation $x^2+y^2=1$ we have that it corresponds to the circle of center $(0,0)$ and radius 1.

Indeed if $P(x,y)$ is a point in the plane, the quantity x^2+y^2 represents its distance from the origin (Pythagorean theorem).

It is easy to see that the circle does not pass the vertical line test, so it is not the graph of a function.



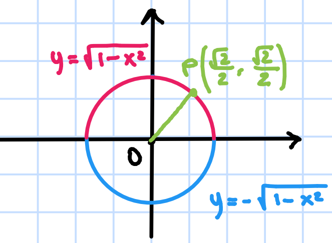
However we can see the circle as the union of the graphs of the previous two functions f_1 and f_2



Now that we have understood the difference between a function explicitely / implicitely defined, let us consider the following problem:

Problem: Find the tangent line to the circle $x^2+y^2=1$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Solution: So far we are only able to take derivatives of explicitely defined functions.



First we have to notice that the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is a point of the graph of the function $f_1(x) = \sqrt{1-x^2}$.

We have:

chain rule

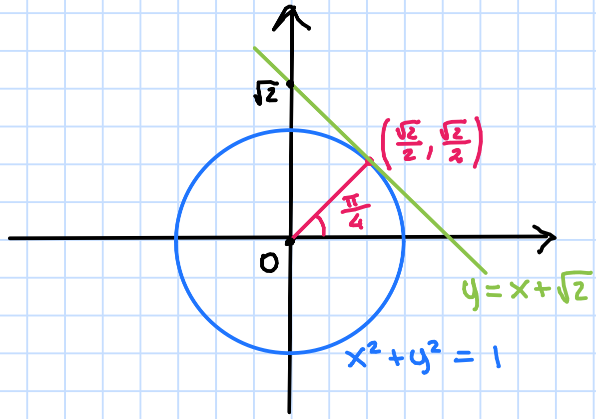
$$f_1'(x) = \left(\sqrt{1-x^2}\right)' = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-2x}{2\sqrt{1-x^2}}.$$

Hence the slope of the tangent line to the graph of f_1 at the point $x = \frac{\sqrt{2}}{2}$ is:

$$f'\left(\frac{\sqrt{2}}{2}\right) = \frac{-2\cdot\frac{\sqrt{2}}{2}}{2\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}} = \frac{-\sqrt{2}}{2\sqrt{1-\frac{2}{4}}} = \frac{-\sqrt{2}}{\cancel{2}\frac{\sqrt{2}}{\cancel{2}}} = -1,$$

and the tangent line has equation:

$$y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$$
$$\Updownarrow$$
$$y = -x + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$
$$\Updownarrow$$
$$y = -x + \sqrt{2}$$



Let us now consider a similar problem with a different curve.

Problem: Find the tangent line to the curve described by the equation:

$$y^3 + xy = \cos(y) - \sin(x)$$

at the point $\left(\frac{\pi}{2}, 0\right)$.

In this case it is not trivial to write y explicitly as a function of x.

Hence for solving the problem we need a new method called IMPLICIT DIFFERENTIATION, that allows us to differentiate y with respect to x, without finding an explicit expression for y.

We will describe the method of implicit differentiation in 4 steps:

(1) Take the derivative of each side of the equation with respect to x (do not forget to treat y as a function of x) and apply the differentiation rules

$$\frac{d}{dx}\left[y^3 + xy\right] = \frac{d}{dx}\left[\cos(y) - \sin(x)\right]$$

$$\Downarrow \text{ SUM RULE}$$

CHAIN RULE (Leibniz)

$$\frac{d}{dx}y^3 = \underbrace{\frac{d}{dy}y^3}_{3y^2} \cdot \frac{dy}{dx}$$

$$\underbrace{\frac{d}{dx}(y^3)}_{\text{chain rule}} + \underbrace{\frac{d}{dx}(xy)}_{\text{product rule}} = \underbrace{\frac{d}{dx}(\cos(y))}_{\text{chain rule}} - \frac{d}{dx}(\sin(x))$$

$$\Downarrow$$

$$3y^2 \cdot \frac{dy}{dx} + \left(\frac{d}{dx}(x)\right) \cdot y + x \cdot \frac{dy}{dx} = -\sin(y) \cdot \frac{dy}{dx} - \cos(x)$$

$$\Downarrow$$

$$3y^2 \cdot \frac{dy}{dx} + 1 \cdot y + x \cdot \frac{dy}{dx} = -\sin(y) \cdot \frac{dy}{dx} - \cos(x)$$

② You have now an ordinary linear equation where the unknown you want to solve for is $\frac{dy}{dx}$. Solve it

Bring all the terms where $\frac{dy}{dx}$ appears on the same side:

$$\underbrace{3y^2 \frac{dy}{dx} + x \frac{dy}{dx} + \sin(y) \frac{dy}{dx}}_{\text{terms with } \frac{dy}{dx}} = \underbrace{-y - \cos(x)}_{\text{terms without } \frac{dy}{dx}}$$

Factor by $\frac{dy}{dx}$:

$$\left(3y^2 + x + \sin(y)\right) \frac{dy}{dx} = -y - \cos(x).$$

Divide by the coefficient of $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{y + \cos(x)}{3y^2 + x + \sin(y)}$$

note that this time the slope of the tangent line depends on both coordinates of the point.

Note that $\frac{dy}{dx}$ represents the slope of the tangent line to the original curve at the generic point (x,y).

③ Substitute the coordinates of the point in order to find the slope of the tangent line at that point.

$$P\left(\frac{\pi}{2}, 0\right) \Longrightarrow \left.\frac{dy}{dx}\right|_{\substack{x=\frac{\pi}{2} \\ y=0}} = -\frac{0 + \cos\left(\frac{\pi}{2}\right)}{3 \cdot 0 + \frac{\pi}{2} + \sin(0)} = \frac{0}{\frac{\pi}{2}} = 0$$

Hence at $P\left(\frac{\pi}{2}, 0\right)$ the tangent line to the curve has slope 0

④ Write an equation of the tangent line

An equation for the tangent line at $P\left(\frac{\pi}{2}, 0\right)$ is $y=0$.

If we go back to the first problem with the circle $x^2+y^2=1$ we see that we get the same result if we apply implicit differentiation:

$$x^2+y^2=1$$
$$\Downarrow$$

① differentiate both side

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(1)$$
$$\Downarrow$$
$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=0$$
$$\Downarrow$$

② solve for $\frac{dy}{dx}$

$$2x+2y\cdot\frac{dy}{dx}=0$$
$$\Downarrow$$
$$2y\,\frac{dy}{dx}=-2x$$
$$\Downarrow$$
$$\frac{dy}{dx}=-\frac{x}{y}$$

③ plug in the coordinates of the point

$$\left.\frac{dy}{dx}\right|_{\substack{x=\frac{\sqrt{2}}{2}\\ y=\frac{\sqrt{2}}{2}}}=-\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=-1$$

④ write the equation of the tangent line

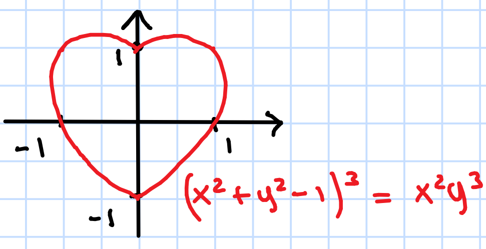
$$y-\frac{\sqrt{2}}{2}=-1\left(x-\frac{\sqrt{2}}{2}\right)$$

Let us make mathematics romantic.

In particular the following equation is very romantic:

$$\left(x^2+y^2-1\right)^3=x^2y^3.$$

Indeed it is the equation of the "heart curve":



We want to find the tangent line to the hearth at the point $(1,1)$.

① DIFFERENTIATE BOTH SIDE

$$\frac{d}{dx}\left(x^2+y^2-1\right)^3 = \frac{d}{dx}x^2y^3$$

$$\Downarrow$$

$$3(x^2+y^2-1)^2\frac{d}{dx}\left(x^2+y^2-1\right) = \left(\frac{d}{dx}(x^2)\right)\cdot y^3 + x^2\cdot\frac{d}{dx}y^3$$

$$\Downarrow$$

$$3(x^2+y^2-1)^2\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)-\frac{d}{dx}(1)\right] = 2xy^3 + x^2\cdot 3y^2\frac{dy}{dx}$$

② SOLVE THE EQUATION FOR $\frac{dy}{dx}$

$$3(x^2+y^2-1)^2\left[2x+2y\frac{dy}{dx}\right] = 2xy^3+x^2 3y^2\frac{dy}{dx}$$

$$\Downarrow$$

$$3(x^2+y^2-1)^2\cdot 2x + 3(x^2+y^2-1)^2\cdot 2y\frac{dy}{dx} = 2xy^3+x^2 3y^2\frac{dy}{dx}$$

$$\Downarrow$$

$$\left[3(x^2+y^2-1)^2 2y - x^2 3y^2\right]\frac{dy}{dx} = -3(x^2+y^2-1)^2 2x + 2xy^3$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{-3(x^2+y^2-1)^2 2x + 2xy^3}{3(x^2+y^2-1)^2 2y - x^2 3y^2}$$

$$\Downarrow$$

③ PLUG IN THE COORDINATES OF THE POINT

slope

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{-3(1+1-1)^2\cdot 2+2}{3(1+1-1)^2\cdot 2-3} = \frac{-6+2}{6-3} = -\frac{4}{3}$$

④ WRITE THE EQUATION OF THE TANGENT LINE

$$y-1 = -\frac{4}{3}(x-1) \Longrightarrow y = -\frac{4}{3}x+\frac{7}{3}$$