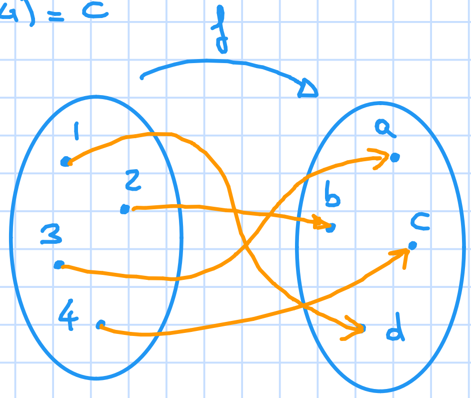
INVERSE FUNCTIONS AND LOGARITHMS (Sec. 3.2)

In order to introduce the logarithmic function we need to recall the notion of the inverse of a function when it exists.

Let us consider the following two functions with domain $\{1,2,3,4\}$ and codomain $\{a,b,c,d\}$.

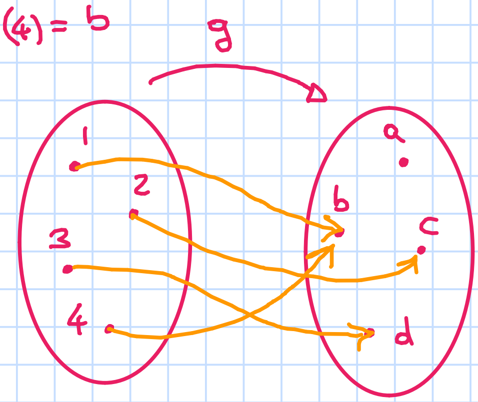
$f: \{1,2,3,4\} \to \{a,b,c,d\}$

$f(1) = d$
$f(2) = b$
$f(3) = a$
$f(4) = c$

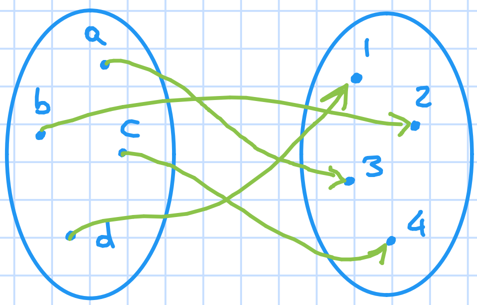


$g: \{1,2,3,4\} \to \{a,b,c,d\}$

$g(1) = b$
$g(2) = c$
$g(3) = d$
$g(4) = b$



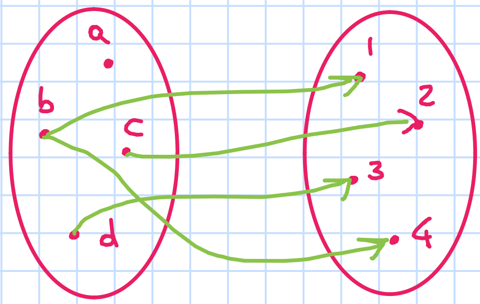
For each of the previous functions f and g, let us see what happens when we exchange the domain with the codomain and we "reverse" the arrows:



This situation corresponds to a new function that we will denote f^{-1} such that:

$f^{-1}: \{a,b,c,d\} \to \{1,2,3,4\}$

$f^{-1}(a) = 3$
$f^{-1}(b) = 2$
$f^{-1}(c) = 4$
$f^{-1}(d) = 1$



This situation does not correspond to a function since the input b gives rise to two outputs 1 and 4.

We say that f possesses an inverse, f^{-1}, while g does not.

If we analyze the situation we remark that for f all the inputs have a different output, while for g there are two different values (1 and 4) with the same output (b):

$$g(1) = g(4) = b.$$

If $g^{-1}(b)$ denotes the set of elements of the domain $\{1,2,3,4\}$ which are sent on b, we write this fact in the following way:

$$g^{-1}(b) = \{1,4\}$$

We say that f is a "one-to-one function" while g is not.

Def: A function f is called one-to-one if it never takes on the same value twice. In formula:

[P ⇒ Q] if $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ different inputs correspond to different outputs

which is equivalent to:

[not Q ⇒ not P] if $f(x_1) = f(x_2) \implies x_1 = x_2$

Remark: If a function $f: A \to B$ is one-to-one, then for every element b in B there exists at most one element in A which is sent on b, i.e.

$f^{-1}(b)$ has at most one element

So, if we go back to the function g, according to the previous definition, it is not one-to-one because $g(1) = g(4)$ but $1 \neq 4$.

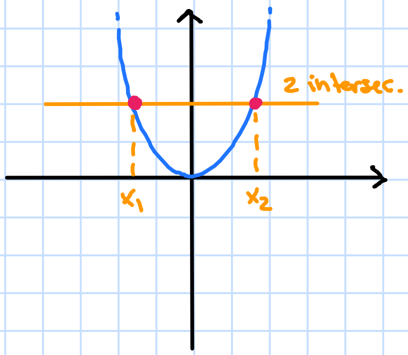
If a function f is defined on $\mathbb{R}$ with values in $\mathbb{R}$, i.e. $f: \mathbb{R} \to \mathbb{R}$, we can use its graph in the plane for determining whether it is one-to-one.

We have indeed the following geometric method:

HORIZONTAL LINE TEST

A function is one-to-one if no horizontal line intersects its graph more than once, or in other words every horizontal line intersects the graph at most once.

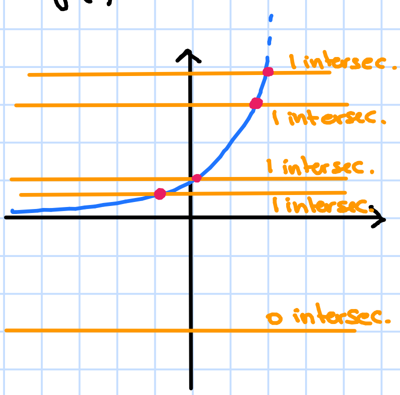
ex: $f(x) = x^2$



✗ no one-to-one

$f(x_1) = f(x_2)$ with $x_1 \neq x_2$

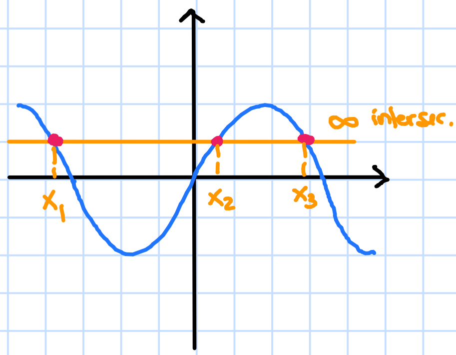
$f(x) = e^x$



✓ one to one

every horizontal line intersects the graph AT MOST once

$f(x) = \sin(x)$



✗ no one-to one

$f(x_1) = f(x_2) = f(x_3) = \dots$ with x_1, x_2, x_3 all different each other

Remark: Note that all strictly increasing or strictly decreasing functions are one-to-one.

For one-to-one functions we can define an inverse function:

Def: Let f be a one-to-one function with domain A and range B.

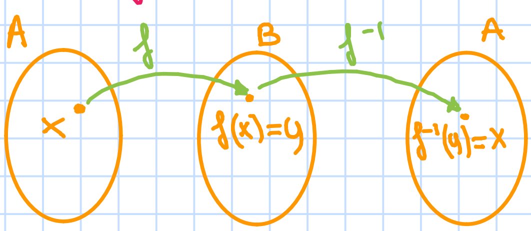
Then its inverse function f^{-1} has domain B and range A and it is defined by:

$$\text{for all } y \text{ in } B, \quad f^{-1}(y) = x \iff f(x) = y.$$

If $f: A \to B$ and $f^{-1}: B \to A$ is its inverse function then f and f^{-1} verify the following cancellation equations:

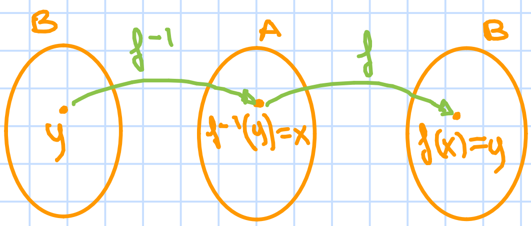
for all x in A

$$f^{-1}(f(x)) = x$$



for all y in B

$$f(f^{-1}(y)) = y$$



Remark: Note that $f^{-1}(x) \neq [f(x)]^{-1}$!!

inverse function of f

reciprocal function of f: $\frac{1}{f(x)}$

ex: $f = e^x$ is a one to one function with domain $\mathbb{R}$ and range $(0,\infty)$:

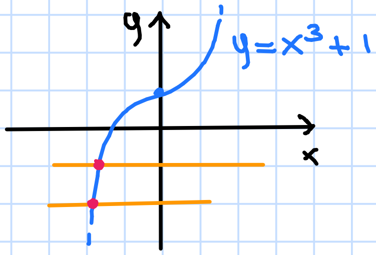
$$f: \mathbb{R} \to (0,\infty)$$
$$0 \mapsto 1$$
$$1 \mapsto e$$
$$\vdots$$
etc.

Hence f possesses an inverse f^{-1} with domain $(0,\infty)$ and range $\mathbb{R}$:

$$f^{-1}: (0,\infty) \to \mathbb{R}$$
$$1 \mapsto 0 \qquad f^{-1}(1) = 0 \text{ since } f(0) = 1$$
$$e \mapsto 1 \qquad f^{-1}(e) = 1 \text{ since } f(1) = e$$

EXERCISE: Find the inverse function of $f(x) = x^3 + 1$.

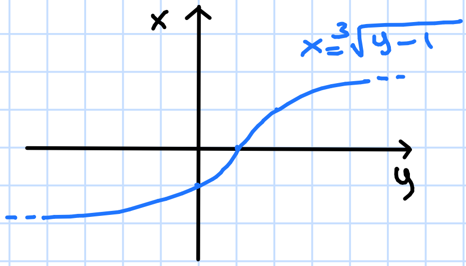
First of all f is one-to-one, since it passes the horizontal line test.



If $f(x) = y$ then we define $f^{-1}(y) = x$.

$$y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = \sqrt[3]{y-1}$$

Then $f^{-1}(y) = x = \sqrt[3]{y-1}$



Remark that this graph is obtained by reflecting the graph of f about the line $y = x$.

Theorem: If f is a one-to-one continuous function defined on an interval, then its inverse f^{-1} is also continuous

LOGARITHMIC FUNCTION

For $a>0$, $a \neq 1$ the exponential function $f(x) = a^x$ is strictly increasing or decreasing. It is then one-to-one.

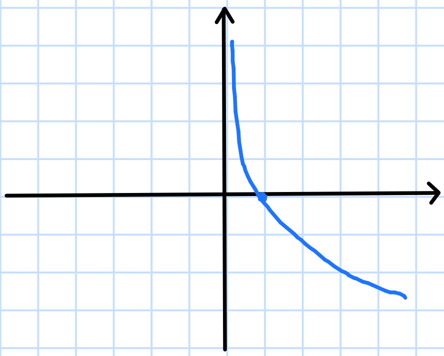
Its inverse f^{-1} is called logarithmic function with base a and denoted $\log_a$.

By definition we have

$$\log_a x = y \iff a^y = x$$

From the properties of the exponential function we can deduce properties for the logarithmic function

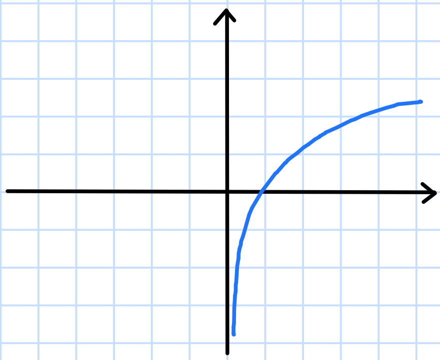
PROPERTIES OF a^x		PROPERTIES OF $\log_a x$
• domain $\mathbb{R}$	$\rightsquigarrow$	range $\mathbb{R}$
• range $(0,\infty)$	$\rightsquigarrow$	domain $(0,\infty)$
• continuous	$\rightsquigarrow$	continuous



$0<a<1$

$\lim_{x \to 0^+} \log_a x = \infty$

$\lim_{x \to \infty} \log_a x = -\infty$



$a>1$

$\lim_{x \to 0^+} \log_a x = -\infty$

$\lim_{x \to \infty} \log_a x = \infty$

Cancellation laws: $f(x) = a^x$, $f^{-1}(x) = \log_a (x)$

For all x in $\mathbb{R}$, $f^{-1}(f(x)) = x \rightsquigarrow \log_a (a^x) = x$, for all x in $\mathbb{R}$

For all x in $(0,\infty)$, $f(f^{-1}(x)) = x \rightsquigarrow a^{\log_a x} = x$, for all $x>0$.

ex: • $\log_{\underset{a}{2}} (\underset{x}{8}) = \underset{y}{3}$ since $\underset{a}{2}^{\overset{y}{3}} = 8$. ← x

• $\log_a 1 = 0$ since $a^0 = 1$, for all $a>0$, $a \neq 1$.

The laws of exponential can be turned in laws of logarithms:

LAWS OF LOGARITHMS

If $a>0$, $a\neq 1$ and $x,y>0$ then

(1) $\log_a(xy) = \log_a(x) + \log_a(y)$

(2) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

(3) $\log_a(x^r) = r\log_a(x)$, where r is any real number

Proof

(1) $$a^{\log_a(x)+\log_a(y)} \underset{a^{x+y}=a^x\cdot a^y}{=} a^{\log_a(x)}\cdot a^{\log_a(y)} \underset{a^{\log_a(x)}=x}{=} x\cdot y \Rightarrow$$

$\Rightarrow$ by definition $\log_a(xy) = \log_a(x) + \log_a(y)$

(2) $$a^{\log_a(x)-\log_a(y)} \underset{a^{x-y}=\frac{a^x}{a^y}}{=} \frac{a^{\log_a(x)}}{a^{\log_a(y)}} \underset{a^{\log_a(x)}=x}{=} \frac{x}{y}$$

$\Rightarrow$ by definition $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

(3) $$a^{r\log_a(x)} \underset{a^{xy}=(a^x)^y}{=} \left(a^{\log_a(x)}\right)^r \underset{a^{\log_a x}=x}{=} x^r$$

$\Rightarrow$ by definition $\log_a(x^r) = r\log_a(x)$

ex: $\log_3 18 - \log_3 2 = \log_3 \frac{18}{2} = \log_3 9 = \log_3 3^2 = 2.$

As in the case of the exponential function, there exists a "most convenient" base for the logarithm function, which is again the number e.

Notation

$\log x = \log_{10} x$

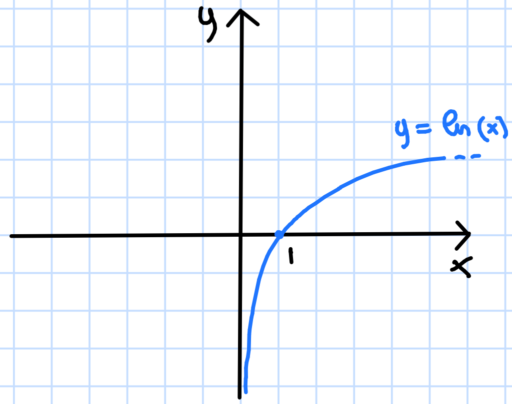
$\ln x = \log_e x$

Natural logarithm

The logarithm function with base e is called natural logarithm and is denoted:

$$f(x) = \ln(x)$$

Properties of $\ln(x)$



- domain $(0, \infty)$
- range $\mathbb{R}$
- continuous on $(0, \infty)$
- $\lim_{x \to 0^+} \ln(x) = -\infty$
- $\lim_{x \to \infty} \ln(x) = \infty$
- for all $x > 0$ $\ln x = y \iff e^y = x$
- $\ln(e^x) = x$ for all x in $\mathbb{R}$
- $e^{\ln(x)} = x$ for all $x > 0$.
- $\ln 1 = 0$
- $\ln e = 1$
- $\log_a x = \dfrac{\ln(x)}{\ln(a)}$