MAXIMUM AND MINIMUM VALUES (Sec. 4.1)

Differential calculus finds some of its most important applications in optimization problems.
Here are two real-life examples:

- What is the shape of a can that minimizes manufacturing costs?
- What is the largest rectangular surface that I can enclose with a given quantity of fencing?

This kind of problems can be reduced to finding the "maximum" and "minimum" values of a function.

We have the following definition:

Def: Let f be a function of domain D and let c be a number in D.

this is a x-value

We say that $f(c)$ is

note that this is a y-value

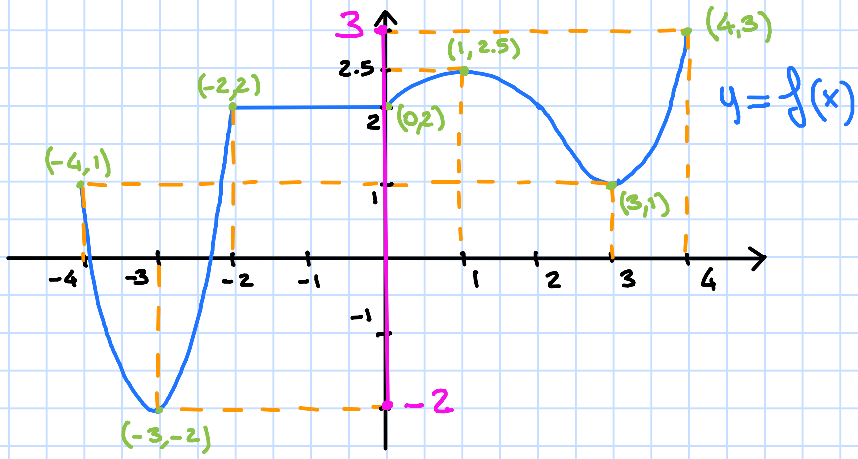
- an absolute (global) maximum value of f if $f(c) \geq f(x)$ for all x in D.
- an absolute (global) minimum value of f if $f(c) \leq f(x)$ for all x in D.
- a relative (local) maximum value of f if $f(c) \geq f(x)$ when x is near c, i.e. if there exists an open interval I, with c in I, such that $f(c) \geq f(x)$ for all x in I.
- a relative (local) minimum value of f if $f(c) \leq f(x)$ when x is near c, i.e. if there exists an open interval I, with c in I, such that $f(c) \leq f(x)$ for all x in I.

If $f(c)$ is a local/absolute maximum/minimum value of f we say that f has a local/absolute maximum/minimum value at c.

Remark:
- It is more common to talk about absolute and local maximum/minimum values rather than global and relative maximum/minimum values.
- If the range of a function f of domain D is given by the close interval $[M,N]$ then M is the absolute minimum value and N is the absolute minimum value of f in D.

Geometrically it is easy to recognize absolute / local maximum / minimum values.
Let us consider the following example.

Graph of a function f on the domain $[-4,4]$



The range of f on the domain $[-4,4]$ is given by the closed interval

range $\rightarrow$ $[-2,3]$

abs. min. value (-2) abs. max. value (3)

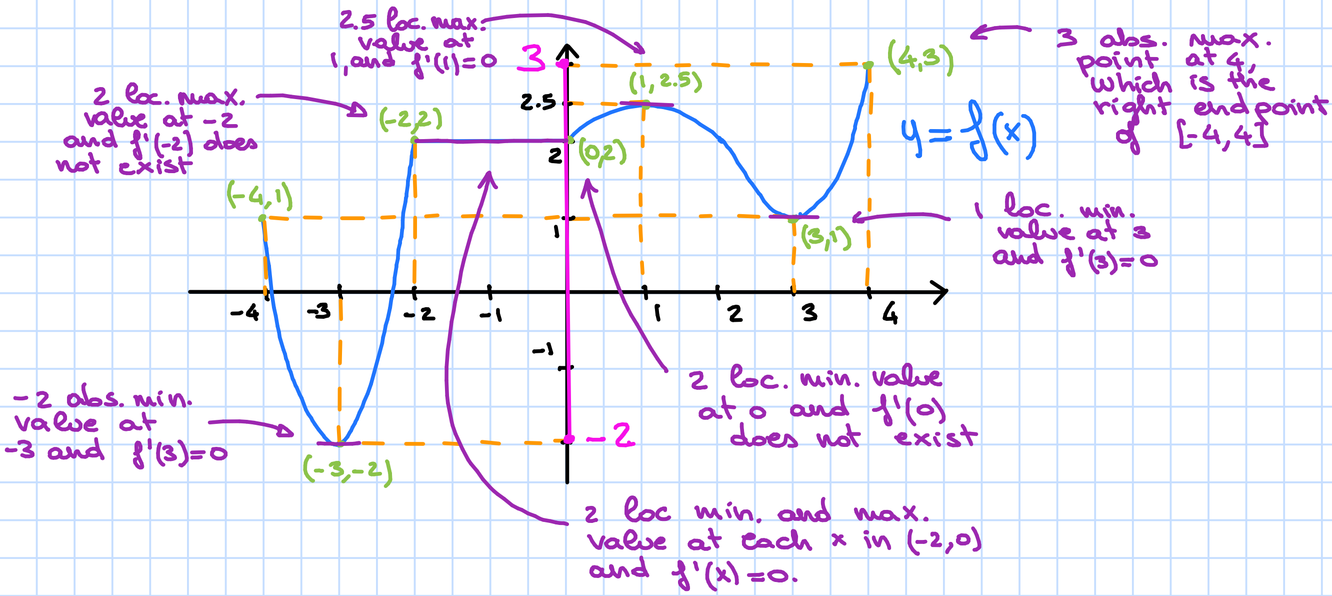
Then -2 ($f(c)$) is the absolute minimum value (attained at -3) (c) and 3 is the absolute maximum value (attained at 4)

Moreover:

- -3 is a local minimum value (attained at -2).
- 2 is a local maximum value (attained at -2 and at each x in $(-2,0)$).
- 2 is a local minimum value (attained at 0 and at each x in $(-2,0)$).
- 2.5 is a local maximum value (attained at 1)
- 1 is a local minimum value (attained at 3).

Since the function is constant on $[-2,0]$, at each x in $(-2,0)$ f has at the same time a local maximum and minimum value.

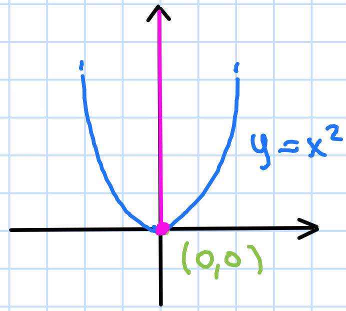
<u>Remark</u>:
- Note that if $f(c)$ is an absolute maximum/minimum then $f(c)$ is also a local maximum/minimum value
- Note also that if $f(c)$ is a local maximum/minimum value, then $f'(c)=0$ or $f'(c)$ does not exist
  $\rightarrow$ this is Fermat's theorem and c is called a critical number (see some pages later...)



ex: ① $f(x) = x^2$

Domain: $\mathbb{R}$, range: $[0, \infty)$

$0 = f(0)$ is the absolute minimum value of f.

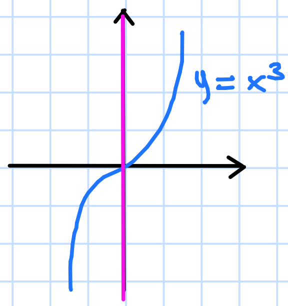


Indeed $f(x) \geq 0$ for all x in $\mathbb{R}$.

Note that f has no absolute maximum value.

② $g(x) = x^3$

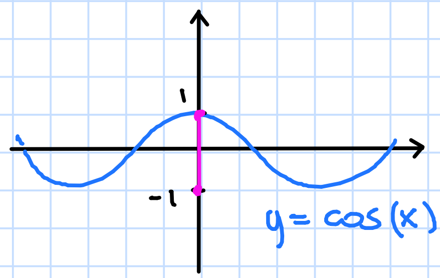
Domain: $\mathbb{R}$, Range: $\mathbb{R}$



g has no absolute minimum neither maximum value.

③ $h(x) = \cos(x)$

Domain: $\mathbb{R}$, Range: $[-1, 1]$.



- 1 is the abs. max. value of h (attained at $x = 2k\pi$, k integer).
- -1 is the abs. min. value of h (attained at $x = \pi + 2k\pi$, k integer).

Indeed for all x in $\mathbb{R}$ we have $-1 \leq f(x) \leq 1$

As we saw in the previous examples, not all functions have an absolute minimum / maximum value.

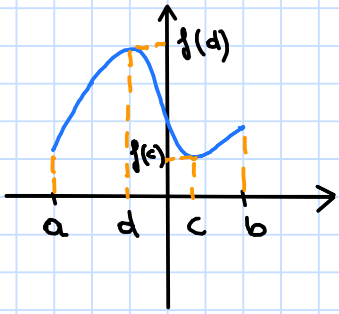
Nevertheless under particular assumptions, the existence of an absolute maximum / minimum value is guaranteed by the EXTREME VALUE THEOREM, known also under the name of Bolzano-Weierstrass' theorem.

EXTREME VALUE THEOREM (Bolzano-Weierstrass theorem)

If f is a continuous function on a closed interval $[a,b]$ then there exist numbers c and d in $[a,b]$ such that

$$f(c) \leq f(x) \leq f(d) \quad \text{for all } x \text{ in } [a,b],$$

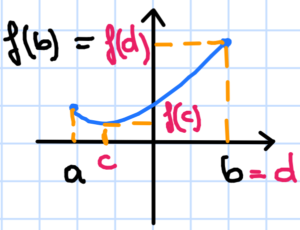
i.e. f attains an absolute minimum value $f(c)$ and an absolute maximum value $f(d)$ at some inputs c, d in $[a,b]$

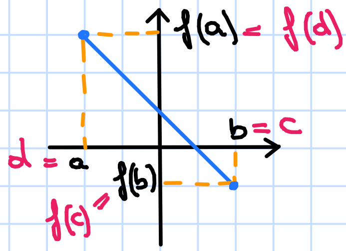


Note that the range of f over $[a,b]$ is $[f(c), f(d)]$.

Remarks: ① Notice that c and d can be found in the interior or among the endpoints of the domain

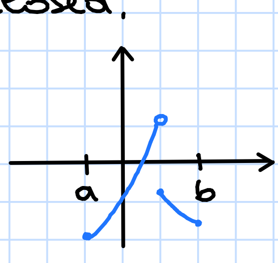
c in (a,b) and d an endpoint of $[a,b]$





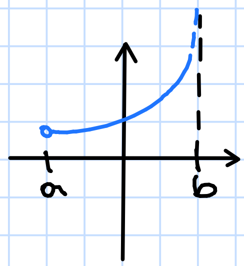
c and d are the endpoints of $[a,b]$

② Extreme value theorem is not true anymore if we remove the hypothesis of continuity or if the interval is not closed.



NON-CONTINUOUS FUNCTION

There is no abs. max. value



OPEN INTERVAL (a,b)

There is no abs. max. neither min. value

range: $(f(a), \infty)$

We notice already that absolute max./min. values have to be found among the values at the endpoints and/or the local max./min. values.

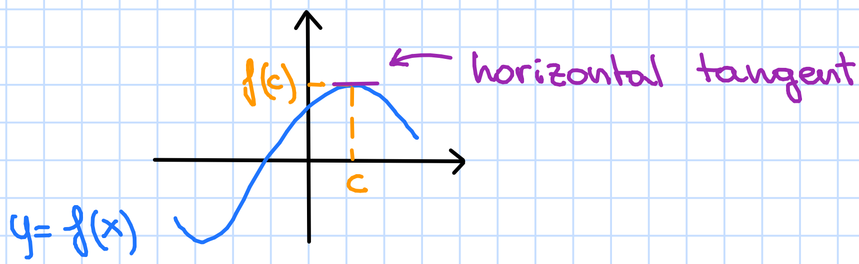
Hence the question now is:

How to find local maximum/minimum values?

If the function f is differentiable, Fermat's theorem represents an answer to this question.

FERMAT'S THEOREM

Let f be a function of domain D and let c be in D. If f has a local maximum or minimum at c and if f is differentiable at c (i.e. $f'(c)$ exists) then $f'(c)=0$.



Proof

We will prove Fermat's theorem in the case where f has a local maximum value at c.

By definition, we have:

$$f(x) \le f(c) \quad \text{when } x \text{ is near } c$$

$$\Updownarrow$$

$$f(x)-f(c) \le 0 \quad \text{when } x \text{ is near } c$$

left-hand limit = right-hand limit

Moreover, since f is differentiable at c, i.e. $f'(c)$ exists, we have:

$$f'(c) = \underbrace{\lim_{x\to c^+} \frac{\overbrace{f(x)-f(c)}^{\le 0}}{\underbrace{x-c}_{>0}}}_{\le 0} = \underbrace{\lim_{x\to c^-} \frac{\overbrace{f(x)-f(c)}^{\le 0}}{\underbrace{x-c}_{<0}}}_{\ge 0}$$

$x>c \Leftrightarrow x-c>0$ $\qquad$ $x<c \Leftrightarrow x-c<0$

$$\Rightarrow \quad 0 \le f'(c) \le 0 \quad \Rightarrow \quad f'(c)=0.$$

Remark: ① The converse of Fermat's theorem is not true, i.e.

if $f'(c) = 0 \not\Rightarrow f$ has a local maximum or minimum value at c.

"does not imply"

counterexample: $f(x) = x^3$

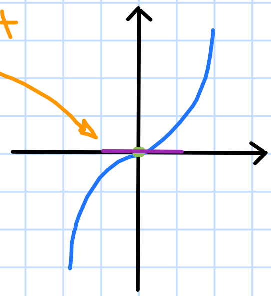
here we look for a function f such that $f'(c)=0$ but that has not a local max or min value at c

$f'(x) = 3x^2$

$c = 0$

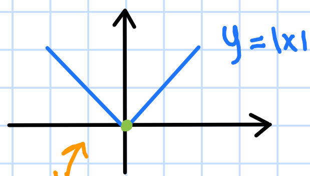
So $f'(0) = 0$, but f has neither a local maximum nor minimum value at 0.

horizontal tangent ($f'(0)=0$) but no local max neither min at 0.



② If f is not differentiable, then it may have local maximum / minimum values also at points c such that $f'(c)$ does not exist.

For example $f(x) = |x|$ has a local minimum value at 0 and $f'(0)$ does not exist



$|x|$ is not differentiable at 0 and $0 = f(0)$ is a local minimum value

This leads us to give a particular name to the points of the domain of a function at which either the derivative is zero or it does not exist.

Def: A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Remark: If f is differentiable, then a critical point is also called a "stationary point". Indeed if we look at f as a position function, then a critical point of f is such that $f'(c) = 0$, that is the velocity at c equals zero.

ex: Find the critical points of the following function:

$$f(x) = e^{-x}\sqrt{x}.$$

Solution

First of all note that the domain of f is given by

$$D = [0, \infty),$$

since $\sqrt{x}$ is defined if and only if $x \geq 0$.

The derivative function of f is:

$$f'(x) \underset{\substack{\uparrow \\ \text{product rule}}}{=} (e^{-x})'\sqrt{x} + e^{-x}(\sqrt{x})' = -e^{-x}\sqrt{x} + e^{-x}\cdot\frac{1}{2\sqrt{x}} =$$

$$= e^{-x}\left(-\sqrt{x} + \frac{1}{2\sqrt{x}}\right) = e^{-x}\cdot\left(\frac{-2x+1}{2\sqrt{x}}\right)$$

Now $c \in D = [0, \infty)$ is a critical number if

- either $f'(c) = 0$. Hence we solve:

$$f'(x) = 0 \iff e^{-x}\left(\frac{-2x+1}{2\sqrt{x}}\right) = 0 \underset{\substack{\uparrow \\ e^{-x} > 0}}{\iff} \frac{-2x+1}{2\sqrt{x}} = 0 \iff$$

$$\underset{\substack{\uparrow \\ \frac{a}{b} = 0 \iff a = 0}}{\iff} -2x+1 = 0 \iff x = \frac{1}{2} \in D.$$

- or $f'(c)$ does not exist: the only point of the domain $D = [0, \infty)$ at which the derivative is not defined is 0 (since it makes the denominator equal 0).

In conclusion the critical points of f are 0 and $\frac{1}{2}$.

Fermat's theorem can be then generalized as follows:

FERMAT'S THEOREM (generalized version)

Let f be a function of domain D and let c be in D. If f has a local maximum or minimum at c then c is a critical number.

If we are given a continuous function defined on a closed interval, the extreme value theorem guarantees that it has an absolute max and min value.

For determining them we can apply "the closed interval method".

CLOSED INTERVAL METHOD

Problem: Find the absolute maximum and minimum value of a continuous function f on a closed interval $[a,b]$.

① Find the critical numbers of f and the corresponding values

② Compute the value of f at the endpoints of the interval $[a,b]$ (i.e. compute $f(a)$ and $f(b)$).

③ Compare the values obtained in step 1 and step 2 and return the absolute maximum and minimum values.

EXAMPLE

Find the absolute maximum and minimum value of the function

$$f(x) = -2x^3 - 3x^2 + 12x + 5 \quad \text{over} \quad [-3,3]$$

① critical numbers and corresponding values.

Since f is a polinomial and thus differentiable, the critical numbers are given only by the numbers c in $(-3,3)$ such that $f'(c)=0$.

$$f'(x) = -6x^2 - 6x + 12 = -6(x^2+x-2) = -6(x+2)(x-1)$$

$$\Rightarrow f'(x)=0 \iff -6(x+2)(x-1)=0 \iff x=-2 \text{ or } x=1.$$

$$f(-2) = -2(-2)^3 - 3(-2)^2 + 12(-2) + 5 = -15$$

$$f(1) = -2-3+12+5 = 12$$

② values at the endpoints

$$f(-3) = -2(-3)^3 - 3(-3)^2 + 12(-3) + 5 = -4$$

$$f(3) = -2(3)^3 - 3(3)^2 + 12 \cdot 3 + 5 = -40$$

③ compare and return

Among the previous values in blue, the lowest is -40 (attained at 3) and the largest is 12 (attained at 1).

Hence the absolute minimum value is -40 and the alsososolute maximum value is 12.

This implies also that the range of the function on $[-3,3]$ is $[-40, 12]$.