THE MEAN VALUE THEOREM (Sec. 4.2)

In this section we will explore two important results of differential calculus:

- Rolle's theorem (Rolle, 1691)
- Mean Value Theorem (Cauchy, 1823).

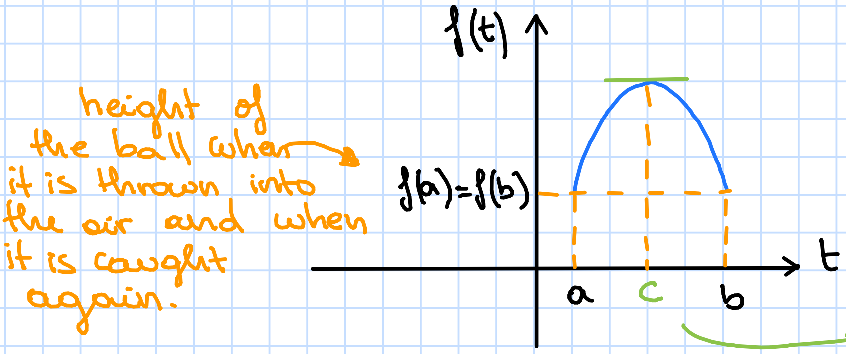
We will see that Rolle's theorem (which is chronologically previous to the Mean Value Theorem) is nothing else than a particular case of the Mean Value Theorem.

Before giving the formal statement of Rolle's theorem, consider the following situation.

Imagine that you throw a ball straight up into the air from a certain height, and then you catch it, at the same height. Beacause of the gravitational acceleration, the initial velocity of the ball will decrease until the ball reaches the peak, where the velocity changes sign and starts increasing again (in absolute value). In particular, the instantaneous velocity of the ball at the 'peak' of its trajectory is equal to zero.

Let $f(t)$ be the position function of the ball which is thrown in the air at a time $t=a$ and caught at a time $t=b$. Then the graph of f is a parabola.

Note that at a time t $f(t)$ represents the height of the ball with respect to the ground floor.



there exists a time $t=c$ in (a,b) such that the instantaneous velocity at $t=c$ is zero, i.e. $f'(c)=0$.
Note that this happens when the ball reaches the peak of its trajectory, i.e. when $f(t)$ attains its maximum value.

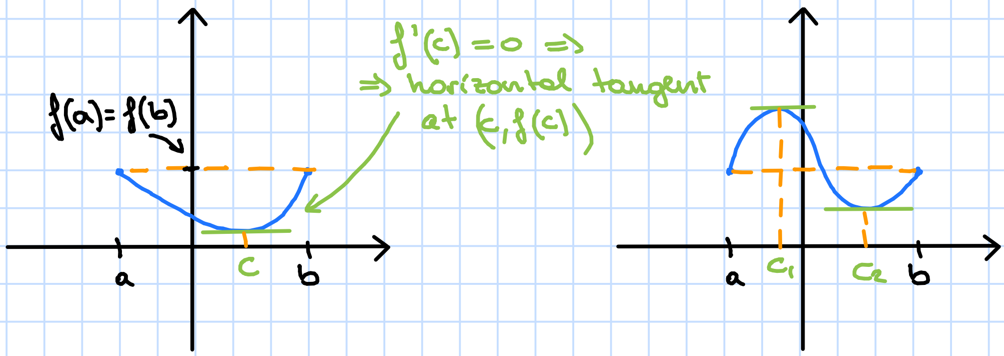
If we make abstraction of the fact that f in our example was a position function, we can state formally the following result.

ROLLE'S THEOREM (Rolle 1691)

Let f be a function which satisfies the following hypothesis:

- f is continuous on the closed interval $[a,b]$,
- f is differentiable on the open interval (a,b),
- $f(a) = f(b)$.

Then there exists c in (a,b) such that $f'(c)=0$.



Note that the tangent line at $(c, f(c))$ is parallel to the secant line through $(a, f(a))$, $(b, f(b))$ ...

By Rolle's theorem we know that there exists at least a number c in (a,b) such that $f'(c)=0$, but there can also exist more numbers in (a,b) with such a property

Proof

Since f is continuous on $[a,b]$, then by the extreme value theorem it has an absolute maximum value and an absolute minimum value over $[a,b]$.

Two different configurations are possible:

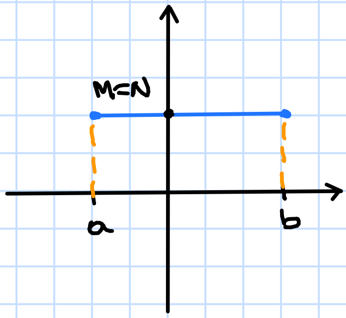
- I CASE

  Both maximum and minimum value are attained at the endpoints of $[a,b]$.

  We can assume that $f(a) = M$ is the abs. max. value and $f(b) = N$ is the abs. min. value. We have $N \leq f(x) \leq M$ for all x in $[a,b]$.

  Since $f(a) = f(b) \Rightarrow M = N \Rightarrow M \leq f(x) \leq M$ for all x in $[a,b] \Rightarrow f(x) = M$ for all x in $[a,b]$, that is f is constant over $[a,b]$.

Thus $f'(x) = 0$ for all x in (a,b) (in particular there exists a number c in (a,b) such that $f'(c)=0$)



• II CASE

At least one among the maximum and minimum value is attained inside (a,b), i.e. f has a local maximum or minimum value at c in (a,b).

Since f is differentiable in (a,b), then $f'(c)$ exists and, by Fermat's theorem, $f'(c) = 0$.

$\square$

The Mean Value theorem generalizes Rolle's theorem.

MEAN VALUE THEOREM (Cauchy 1823)

Let f be a function which satisfies the following hypothesis:

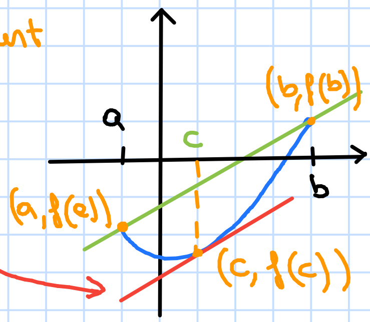
- f is continuous on the closed interval $[a,b]$,
- f is differentiable on the open interval (a,b).

Then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

slope of the tangent line at $(c, f(c))$

slope of the secant line through $(a, f(a))$, $(b, f(b))$.



same slope ⇕ parallel lines

Remarks: ① From a geometrical point of view, the Mean Value Theorem is saying that there exists a number c in (a,b) such that the tangent line to the graph of f at $(c, f(c))$ is parallel (= same slope) to the secant line through $(a, f(a))$ and $(b, f(b))$.

② Rolle's theorem is a particular case of the Mean Value theorem. Indeed, if we add the hypothesis $f(a)=f(b)$, the Mean Value Theorem states that there exists c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \underset{f(a)=f(b)}{=} \frac{0}{b-a} = 0.$$

③ In the case where $f(t)$ is a position function over an interval of time $[a,b]$, the Mean Value Theorem states that there is a time c in (a,b) at which the instantaneous velocity $(f'(c))$ equals the average velocity $\left(\frac{f(b)-f(a)}{b-a}\right)$.

This gives also a "justification" to the name of the theorem.

ex: $f(t) = t^3 - t$ over $[0,2]$

AVERAGE VELOCITY OVER $[0,2]$: $\frac{f(2)-f(0)}{2-0} = \frac{6-0}{2} = 3.$

By the Mean Value Theorem there exists a time c in $(0,2)$ such that the instantaneous velocity at c equals 3. We have:

INSTANTANEOUS VELOCITY AT c: $f'(c) = 3c^2 - 1$

$\Rightarrow f'(c) = 3 \Leftrightarrow 3c^2 - 1 = 3 \Leftrightarrow c^2 = \frac{2}{3} \Leftrightarrow$

$\Leftrightarrow \quad c = \pm\sqrt{\frac{2}{3}}$ (note that $-\sqrt{\frac{2}{3}}$ is outside $(0,2)$)

Hence $c = \sqrt{\frac{2}{3}} \in (0,2)$ is such that $f'(c) = 3$.

Thanks to the Mean Value Theorem, we can recover some information about a function from information about its derivative.

EXERCISES

① Suppose that $f(1) = 1$ and $f'(x) \geq 2$ for all $x > 0$. How small can be $f(3)$?

Solution

before applying a theorem, check always that its hypothesis are satisfied!

We have that f is differentiable on $(0,\infty)$ which implies that f is continuous on $[1,3]$ and differentiable on $(1,3)$.

By the Mean Value Theorem, there exists c in $(1,3)$ such that:

$$f'(c) = \frac{f(3)-f(1)}{3-1} = \frac{f(3)-1}{2}$$

Since $f'(x) \geq 2$ for all x in $(1,3)$, this implies:

$$\frac{f(3)-1}{2} \geq 2 \iff f(3)-1 \geq 4 \iff f(3) \geq 5$$

solve the inequality!
Recall that if you multiply by a negative number, you have to change the sign of the inequality.

Thus $f(3) \geq 5$, that is, the lowest value for $f(3)$ is 5.

② Does there exist a function such that $f(-1) = 3$, $f(2) = 4$ and $f'(x) \geq \frac{1}{2}$ for all x in $\mathbb{R}$?

Solution

The function f is differentiable for all x in $\mathbb{R}$ (since $f'(x)$ exists).

conditions

In particular this implies that f is continuous on $[-1,2]$ and differentiable on $(-1,2)$.

By the Mean Value Theorem, there exists a number c in $(-1,2)$ such that:

$$f'(c) = \frac{f(2)-f(-1)}{2-(-1)} = \frac{4-3}{3} = \frac{1}{3}.$$

But this is in contr iction with the hypothesis that $f'(x) \geq \frac{1}{2}$ for all x (s $\frac{1}{3} < \frac{1}{2}$).
Thus such a function can exist.

The following result is a consequence of the Mean Value Theorem:

Proposition: If $f'(x) = 0$ for all x in (a,b) then $f(x)$ is constant over (a,b).

Proof:
In order to show that f is constant over the interval (a,b) it is enough to prove that wherever we choose x_1 and x_2 in (a,b) we have $f(x_1) = f(x_2)$.

Let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Notice that the interval (x_1, x_2) is contained in (a,b).

Since f is differentiable on (a,b) then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

Then, by the Mean Value Theorem, there exists c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Since by hypothesis $f'(x) = 0$ for all x in (a,b) then we have:

$$\underset{\substack{c \in (a,b) \\ \Downarrow \\ f'(c)=0}}{0} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \underset{x_1 \neq x_2}{\Longrightarrow} f(x_2) - f(x_1) = 0 \Rightarrow f(x_1) = f(x_2).$$

$\square$

<u>Corollary</u>: If f and g are two functions such that $f'(x) = g'(x)$ for all x in an interval (a,b), then for all x in (a,b) $f(x) = g(x) + c$, where c is a constant.

<u>Proof</u>:

Let us consider the function

$$h(x) = f(x) - g(x).$$

For all x in (a,b) we have $h'(x) = f'(x) - g'(x) \overset{\substack{\text{HYPOTHESIS} \\ f'(x)=g'(x) \text{ for all } x \text{ in } (a,b)}}{=} 0$.

By the previous proposition we get that $h(x)$ is constant over (a,b), i.e. there exists a real number c such that

$$h(x) = c \iff f(x) - g(x) = c \iff f(x) = g(x) + c,$$

for all x in (a,b).

$\square$