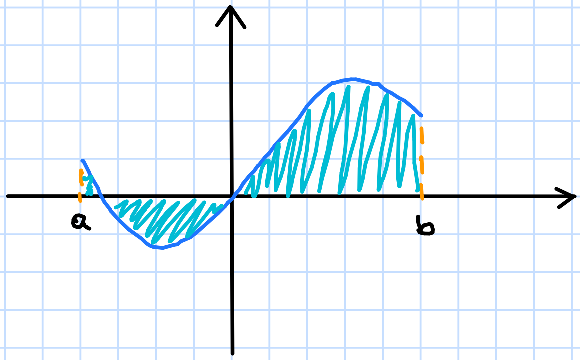
THE FUNDAMENTAL THEOREM OF CALCULUS (Sec. 5.3, 5.4)

Recall from the previous class that if $f(x)$ is a function defined on $[a,b]$ then the number:

$$\int_a^b f(x)\,dx$$

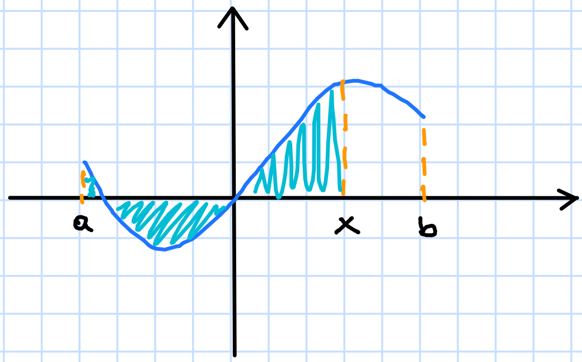
denotes the "area" of the region between the lines $x=a$, $x=b$, the x-axis and the graph of the function.



If now x is a number between a and b, we can consider the function

$$g(x) = \int_a^x f(t)\,dt$$

that at every x in $[a,b]$ associate the "area" of the region represented in the following graph:

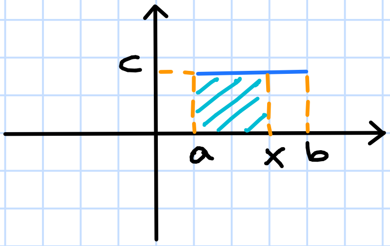


Note that:

- $g(a) = \int_a^a f(t)\,dt = 0$
- $g(b) = \int_a^b f(t)\,dt$
- if $f(x) \geq 0$ for all x in $[a,b]$, then $g(x) \geq 0$.

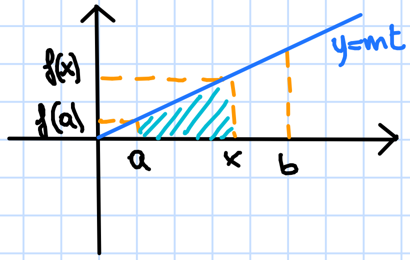
Let us consider the function $g(x)$ when f is a constant or a linear function.

(1) $g(x) = \int_a^x c \, dt$, where c is a constant ($f(x) = c$)



$$\Rightarrow \quad g(x) = (x-a) \cdot c = cx - ac$$

(2) $g(x) = \int_a^x mt \, dt$, where m is a real number ($f(x) = mx$)



$$\Rightarrow \quad g(x) = \frac{1}{2} x f(x) - \frac{1}{2} a f(a) =$$

difference of area of triangles

$$= \frac{1}{2} x \cdot mx - \frac{1}{2} ama =$$

$$= \frac{m}{2} x^2 - \frac{m}{2} a^2$$

→ Note that in both cases the obtained function $g(x)$ is an antiderivative of the function f, i.e. $g'(x) = f(x)$.

(1) $g(x) = cx - ac \Rightarrow g'(x) = c = f(x)$

(2) $g(x) = \frac{m}{2} x^2 - \frac{m}{2} a^2 \Rightarrow g'(x) = mx = f(x)$

This is not a coincidence, but it is called the fundamental theorem of calculus, which establishes a connection between the two branches of calculus: differential calculus and integral calculus.

FUNDAMENTAL THEOREM OF CALCULUS

Suppose f is continuous on $[a,b]$.

PART 1 : If $g(x) = \int_a^x f(t) \, dt$, then $g(x)$ is differentiable on (a,b) and $g'(x) = f(x)$, i.e. $g(x)$ is an antiderivative of $f(x)$.

PART 2 : $\int_a^b f(x) \, dx = F(b) - F(a)$, where F is any antiderivative of f (i.e. $F'(x) = f(x)$).

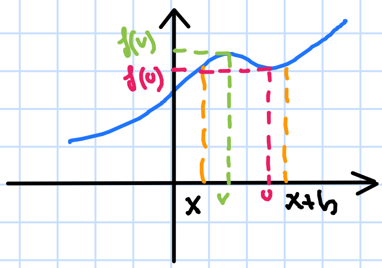
Proof of PART I

We use the definition of derivative:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\overbrace{\int_a^{x+h} f(t)\,dt}^{\int_a^{x} f(t)\,dt+\int_x^{x+h} f(t)\,dt} - \int_a^{x} f(t)\,dt}{h} =$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} \qquad \circledast$$

Assume $h>0$ (the proof is analogous for $h<0$).
Since f is continuous on $[x,x+h]$ (note that $[x,x+h]\subseteq[a,b]$), then, by the extreme value theorem, f has an absolute maximum value $f(v)$ and an absolute minimum value $f(u)$ with $u,v\in[x,x+h]$.



$$f(u)\le f(t)\le f(v) \quad \text{for all } t \text{ in } [x,x+h]$$

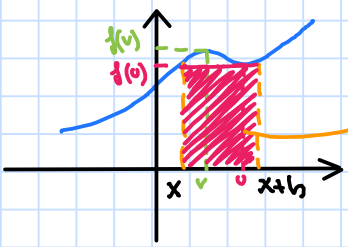
$$\Downarrow \text{ property (8)/(9) of the definite integral}$$

$$\int_x^{x+h}\underbrace{f(u)}_{\text{constant}}\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h}\underbrace{f(v)}_{\text{constant}}\,dt$$

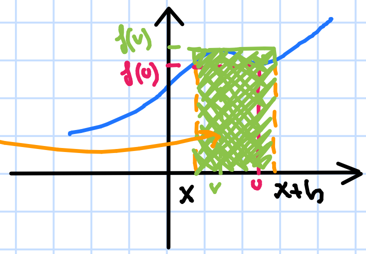
$$\Downarrow$$

$$f(u)\cdot(x+h-x)\le\int_x^{x+h} f(t)\,dt\le f(v)(x+h-x)$$

$$\Downarrow$$



$$\underbrace{f(u)\cdot h}\le\int_x^{x+h} f(t)\,dt\le\underbrace{f(v)\cdot h}$$



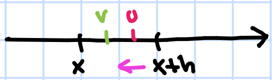
$$\Downarrow \text{ divide by } h \ (h>0)$$

$$f(u)\le\frac{\int_x^{x+h} f(t)\,dt}{h}\le f(v)$$

$$\Downarrow$$

$$\underbrace{\lim_{h\to 0} f(u)}_{\overset{\circledast}{=}\, f(x)} \le \underbrace{\lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}}_{\overset{\circledast}{=}\, g'(x)} \le \underbrace{\lim_{h\to 0} f(v)}_{\overset{\circledast}{=}\, f(x)} \xRightarrow{\text{SQUEEZE THEOREM}} g'(x)=f(x)$$

$\circledast$



as h approaches 0 then v and u approach x and

$f(u)\xrightarrow{h\to 0} f(x)$

$f(v)\xrightarrow{h\to 0} f(x)$

here you need to use the hypothesis of continuity of f

Proof of PART 2 using PART 1

Let F be an antiderivative of f.

By part 1 of the Fundamental theorem of calculus, we know that also $g(x) = \int_a^x f(t)\,dt$ is an antiderivative of f, that is:

$$g'(x) = F'(x) \quad \text{for all } x \text{ in } (a,b)$$

$$\Downarrow$$

there exists c in $\mathbb{R}$ such that $g(x) = F(x) + c$ for all x in $[a,b]$.

Since $g(a) = \int_a^a f(t)\,dt = 0$ we have

$$\underbrace{g(a)}_{=0} = F(a) + c \implies c = -F(a)$$

Therefore $g(x) = F(x) - F(a)$ and

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = g(b) = F(b) - F(a).$$

Remarks

- Part 1 of the Fundamental Theorem of calculus relates differentiation and integration, showing that these two operations are essentially inverses of one another:

DIFFERENTIATION

$$\int_a^x f(t)\,dt = f(x)$$

INTEGRATION

- Part 2 of the Fundamental Theorem of Calculus, also called evaluation theorem, gives a practical method for evaluating integrals. It states that the integral of a function f over some interval can be computed by using anyone of its infinitely many antiderivatives.

EXERCISES

(1) $$\int_0^3 e^x\,dx = \Big[\underbrace{e^x}_{\text{ANTIDERIVATIVE}}\Big]_0^3 = e^3 - e^0 = \underbrace{e^3-1}_{\text{this is a number!}}$$

Note that the result does not depend on the antiderivative choosen:

$$\int_0^3 e^x\,dx = \Big[\underbrace{e^x + c}_{\text{most general antiderivative}}\Big]_0^3 = e^3 + c - (e^0 + c) = e^3 - e^0 = e^3 - 1$$

(2) $$\int_0^{\frac{\pi}{2}} \cos(x) + 1\,dx = \Big[\underbrace{\sin(x) + x}_{\text{antiderivative}}\Big]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2} - (\sin(0) + 0) =$$

$$= 1 + \frac{\pi}{2} - (0+0) = 1 + \frac{\pi}{2}.$$

(3) $$\int_1^e \frac{1}{x} + 2x\,dx = \Big[\ln|x| + x^2\Big]_1^e = \ln|e| + e^2 - (\ln|1| + 1) =$$

$$= 1 + e^2 - (0+1) = e^2.$$

(4) $$\int_1^2 \frac{x^5 + 6\sqrt{x} - 1}{x^2}\,dx = \int_1^2 \frac{x^5}{x^2} + \frac{6\sqrt{x}}{x^2} - \frac{1}{x^2}\,dx =$$

$$= \int_1^2 x^3 + 6x^{\frac{1}{2}-2} - x^{-2}\,dx = \int_1^2 x^3 + 6x^{-\frac{3}{2}} - x^{-2}\,dx =$$

$$= \left[\frac{1}{4}x^4 + \frac{6}{-\frac{3}{2}+1}x^{-\frac{3}{2}+1} - \frac{1}{-2+1}x^{-2+1}\right]_1^2 =$$

$$= \left[\frac{1}{4}x^4 - 12x^{-\frac{1}{2}} + x^{-1}\right]_1^2 = \frac{1}{4}(2)^4 - 12\cdot 2^{-\frac{1}{2}} + 2^{-1} - \left(\frac{1}{4} - 12 + 1\right) =$$

$$= \frac{1}{4}\cdot 16 - \frac{12}{\sqrt{2}} + \frac{1}{2} - \left(-\frac{43}{4}\right) = \frac{16 - 12\cdot 2\sqrt{2} + 2 + 43}{4} =$$

$$= \frac{61 - 24\sqrt{2}}{4}.$$

(5) Compute $\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt$.

Solution

Since $\sqrt{1+t^2}$ is continuous everywhere, then FTC (Part 1) applies and

$$\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt = \sqrt{1+x^2}$$

(6) Let $g(x) = \int_0^{\sin(x)} \sqrt{1+t^2}\,dt$. Compute $g'(x)$.

solution:

We set:

$$f(x) = \sin(x) \quad \text{(INSIDE FUNCTION)} \Rightarrow f'(x) = \cos(x)$$

$$h(x) = \int_0^x \sqrt{1+t^2}\,dt \quad \text{(OUTSIDE FUNCTION)} \Rightarrow h'(x) \underset{\substack{\uparrow \\ \sqrt{1+t^2}\text{ continuous} \\ \Rightarrow \text{FTC}}}{=} \sqrt{1+x^2}$$

We have:

$$g(x) = h(f(x))$$

$$\Downarrow$$

$$g'(x) = [h(f(x))]' \underset{\substack{\uparrow \\ \text{chain rule}}}{=} h'(f(x))\,f'(x) = \sqrt{1+(\sin(x))^2}\cdot\cos(x).$$

(7) Compute $\frac{d}{dx}\int_{\frac{1}{x}}^{1} \arctan(t)\,dt$.

Solution

Let $g(x) = \int_{\frac{1}{x}}^{1} \arctan(t)\,dt = -\int_1^{\frac{1}{x}} \arctan(t)\,dt$.

- $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$
- $h(x) = -\int_1^x \arctan(t)\,dt \Rightarrow h'(x) = \left(-\int_1^x \arctan(t)\,dt\right)' =$

$$= -\left(\int_1^x \arctan(t)\,dt\right)' \underset{\substack{\uparrow \\ \arctan(t)\text{ continuous} \\ \Rightarrow \text{FTC}}}{=} -\arctan(x)$$

Since $g(x) = h(f(x))$ we have:

$$g'(x) = [h(f(x))]' = h'(f(x)) \cdot f'(x) = -\arctan\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right).$$

INDEFINITE INTEGRAL

The indefinite integral is nothing else than a convenient notation for the most general antiderivative, justified by the Fundamental theorem of calculus.

Def: The indefinite integral of a function f, denoted by

$$\int f(x)\,dx,$$

is the most general antiderivative of f, i.e.

$$\int f(x)\,dx = F(x) + c, \quad \text{where} \quad F'(x) = f(x).$$

Since indefinite integrals are antiderivatives, they satisfy the following properties.

PROPERTIES

- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int c f(x)\,dx = c \int f(x)\,dx$

TABLE OF INDEFINITE INTEGRALS

- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c, \quad n \neq -1$
- $\int \frac{1}{x}\,dx = \ln|x| + c$
- $\int e^x\,dx = e^x + c$
- $\int \sin(x)\,dx = -\cos(x) + c$
- $\int \cos(x)\,dx = \sin(x) + c$
- $\int \sec^2(x)\,dx = \tan(x) + c$
- $\int \frac{1}{x^2+1}\,dx = \arctan(x) + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$

EXERCISE : Compute the indefinite integral

$$\int 5\sin(x) + 2\sec^2(x) + \frac{x-3}{x} \, dx.$$

Solution

We have:

$$\int 5\sin(x) + 2\sec^2(x) + \frac{x-3}{x} \, dx = \int 5\sin(x) + 2\sec^2(x) + 1 - \frac{3}{x} \, dx =$$

$$= \int 5\sin(x) \, dx + \int 2\sec^2(x) \, dx + \int 1 \, dx + \int \frac{-3}{x} \, dx =$$

$$= 5\int \sin(x) \, dx + 2\int \sec^2(x) \, dx + \int 1 \, dx - 3\int \frac{1}{x} \, dx =$$

$$= -5\cos(x) + 2\tan(x) + x - 3\ln|x| + C.$$