Calculus I - MAC 2311 - Section 003

Quiz 5 - Solutions

10/24/2018

1) a) [1.5 points] Give the definition of a critical number of a function f.

A critical number *of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

b) [1.5 points] State the Mean Value Theorem.

Let f be a function which is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

2) [4 points] Find the absolute maximum and minimum values of the function

$$f(x) = e^{x^3 - 3x}$$

on the closed interval $[0, 2]$.

Solution:

Since f is a continuous function on the closed interval $[0, 2]$, the Extreme Value Theorem guarantees that f attains an absolute maximum value and an absolute minimum value on $[0, 2]$. Let us find them!

- *Compute the values of f at the endpoints of the interval $[0, 2]$.*

  We have $f(0) = e^0 = 1$ and $f(2) = e^{8-6} = e^2 \sim 7.39$.

- *Find the critical numbers of f in $(0, 2)$ and their corresponding values.*

  Since f is differentiable everywhere, its critical numbers in $(0, 2)$ are all the numbers c in $(0, 2)$ such that $f'(c) = 0$.
  Here we have:

$$f'(x) = (e^{x^3-3x})' = e^{x^3-3x} \cdot (3x^2 - 3) = e^{x^3-3x} \cdot 3(x^2 - 1) = e^{x^3-3x} \cdot 3(x - 1)(x + 1).$$

  Thus $f'(x) = 0$ if and only if $x = -1$ or $x = 1$ (note that $e^{x^3-3x} \neq 0$ for all x). Now, only 1 is inside the interval $(0, 2)$ and the corresponding value is $f(1) = e^{1-3} = e^{-2} \sim 0.14$.

- *Compare the values obtained in step 1 and step 2 and return the absolute maximum and the absolute minimum values of f.*

  We have $f(1) = e^{-2} < f(0) = 1 < f(2) = e^2$, so the absolute maximum value of f on $[0, 2]$ is e^2 and the absolute minimum value of f on $[0, 2]$ is e^{-2}.

3) [4 points] Let f be a differentiable function such that $f'(x) \leq -1$ for all x in $\mathbb{R}$. If $f(0) = -2$, what is the lowest value that f may attain at -2?

Solution:

We consider the function f on the closed interval $[-2, 0]$. Since f is a function which is differentiable everywhere, we have in particular that f is continuous on $[-2, 0]$ and differentiable on $(-2,0)$. Thus, by the Mean Value Theorem, there exists c in $(-2,0)$ such that

$$f'(c) = \frac{f(0) - f(-2)}{0 - (-2)} \overset{f(0)=-2}{=} \frac{-2 - f(-2)}{2}.$$

By hypothesis $f'(c) \leq -1$. Therefore we have:

$$\frac{-2 - f(-2)}{2} \leq -1 \Leftrightarrow -2 - f(-2) \leq -2 \Leftrightarrow f(-2) \geq 0.$$

In conclusion, the lowest value that f may attain at -2 is 0.

4) Is the following statement true or false? Justify fully your answer.

Let f be a continuous function. If f has a local minimum at $x = 2$, then $f'(2) = 0$.

FALSE

By the Generalized Fermat's Theorem, if f has a local minimum at $x = -2$, then $x = -2$ is a critical number, i.e. either $f'(2) = 0$ or $f'(2)$ does not exist. Then a counterexample to our statement would be a function which has a local minimum at $x = -2$ but is not differentiable at $x = -2$.

If we consider $f(x) = |x + 2|$, we have that f is continuous at $x = -2$, but not differentiable at $x = -2$, and $f'(x) < 0$ on $(-\infty, -2)$ and $f'(x) > 0$ on $(-2, \infty)$. This implies that f has a local minimum at $x = -2$ and $f'(-2)$ is not defined.