CONTINUITY (Sec. 1.5 of the book)

In the previous class we saw that for most of the functions, if we want to compute the limit at a number in the domain we have just to evaluate the function at that number.

Indeed a polynomial, a rational function, trigonometric functions, etc. are "continuous" at each point of their domain.

Def: A function is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

A function is continuous on an interval (c,d) if it is continuous at every point of the interval.

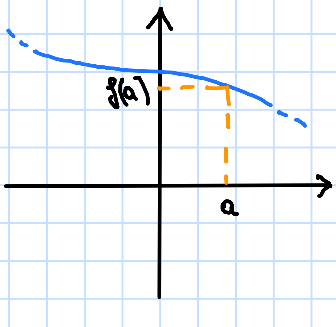
Remarks:

- The condition $\lim_{x\to a} f(x) = f(a)$ is equivalent to all the following three ones:

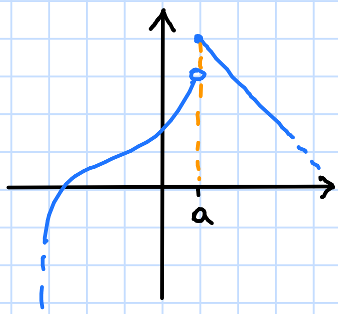
$$\begin{cases} 1)\ f(a) \text{ is defined (i.e. } a \text{ is in the domain of the function)} \\ 2)\ \lim_{x\to a} f(x) \text{ exists, i.e. } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \underbrace{L}_{\text{a number (not } \pm\infty)} \\ 3)\ L = f(a) \end{cases}$$

- Roughly speaking, if f is a continuous function on an interval I then sufficiently small changes in x (in I) result in arbitrarily small changes in $f(x)$.

  Visually the graph of a continuous function has no "breaks" (no holes, no jumps) and I can draw it without detaching my pencil from the paper (again, this is very roughly speaking)



CONTINUOUS



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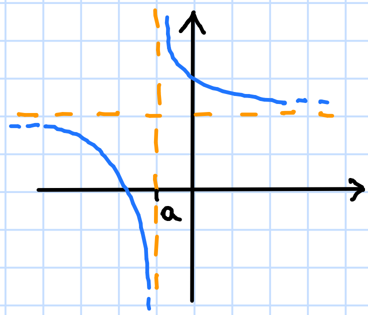


NOT CONTINUOUS

- Most of physical phenomena are continuous:
  - the displacement of a car along the road is a continuous function of time (otherwise the car would have teleported)
  - the temperature in a room is a continuous function of time
  - a person's height is a continuous function of time

but

the function of electric current is not always continuous (imagine what happens when you press a switch...)

Indeed a continuous process is one that takes place gradually, without interruption or abrupt change.

<u>Example</u>: Any polynomial, $\sin(x)$, $\cos(x)$ are continuous everywhere, that is for all $x \in \mathbb{R}$.

It is also natural to define when a function is not continuous at a number a, that is when it is "discontinuous" at a.

<u>Def</u>: Let f be a function defined near a, except possibly at a.
The function f is said to be discontinuous at a if f is not continuous at a.
In this case we say that a is a discontinuity for f.

<u>Remark</u>: Note that a number a can be a discontinuity for f even if it is not in its domain (but f has to be defined near a).

There exist three kinds of discontinuity:

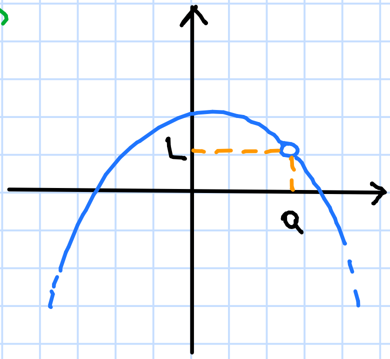
① REMOVABLE DISCONTINUITY

<u>Def</u>: A number a is said to be a removable discontinuity for f if

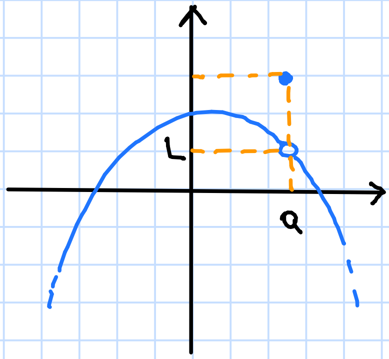
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \quad \leftarrow L \text{ a number (not } \pm\infty)$$

and either a does not belong to the domain of f or $f(a) \neq L$.

Removable discontinuity



a does not belong to the domain



$f(a) \neq L$

ex: • $f_1(x) = \dfrac{x^3 - x^2}{x-1}$

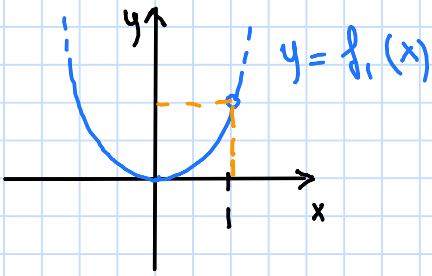
We study the continuity of f_1 at $x=1$ (indeed f_1 is continuous for all $x \neq 1$)!

$$\lim_{x \to 1^-} \frac{x^3 - x^2}{x-1} = \lim_{x \to 1^-} \frac{x^2\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1^-} x^2 = 1$$

$$\lim_{x \to 1^+} \frac{x^3 - x^2}{x-1} = \lim_{x \to 1^+} \frac{x^2\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1^+} x^2 = 1$$

Then $\lim_{x \to 1^-} f_1(x) = \lim_{x \to 1^+} f_1(x)$ and f_1 is not defined at $x=1$.

Hence $x=1$ is a removable discontinuity.



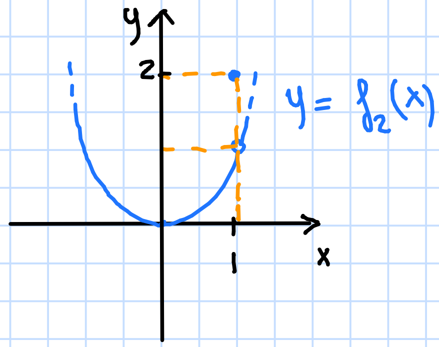
$f_1(x)$ is equal to the function x^2 for all $x \neq 1$. Thus its graph is a parabola with a hole

• $f_2(x) = \begin{cases} \dfrac{x^3 - x^2}{x-1}, & \text{when } x \neq 1 \\ 2, & \text{when } x = 1 \end{cases}$

Again $\lim_{x \to 1^-} f_2(x) = \lim_{x \to 1^+} f_2(x) = 1$ but $f_2(1) = 2$

$\neq$

Hence $x=1$ is a removable discontinuity also for $f_2(x)$.



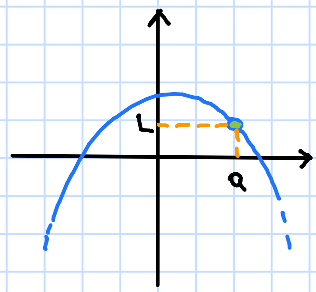
<u>Remark</u>: if a function f has a removable discontinuity at $x=a$, then it is easy to define from f a function which agrees with f for all $x \neq a$ and is continuous at $x=a$
(<u>this explains the adjective removable</u>...)

Roughly speaking, all we have to do is filling the hole!

So, imagine that $\lim_{x \to a} f(x) = L$ with f not defined at a or $f(a) \neq L$. Then the function:

$$g(x) = \begin{cases} f(x), & \text{when } x \neq a \\ L, & \text{when } x = a \end{cases}$$

is continuous at $x=a$



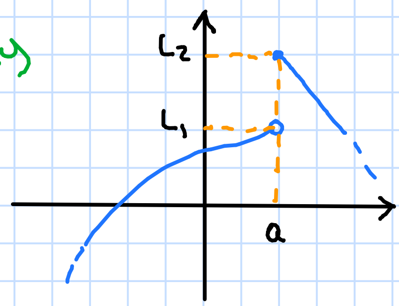
g agrees with f for all $x \neq a$ and $g(a) = \lim_{x \to a} f(x)$

② JUMP DISCONTINUITY

<u>Def</u>: A number a is said to be a jump discontinuity for f if both left-hand and right-hand limits exist (not $\pm\infty$) and

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

Jump discontinuity



$\lim_{x \to a^-} f(x) = L_1 < \infty$

$\lim_{x \to a^+} f(x) = L_2 < \infty$

and $L_1 \neq L_2$

ex: Let us consider the following piecewise function:

$$f(x) = \begin{cases} \cos(x)+1, & \text{when } x \leq 0 \\ -x+1, & \text{when } x > 0 \end{cases}$$

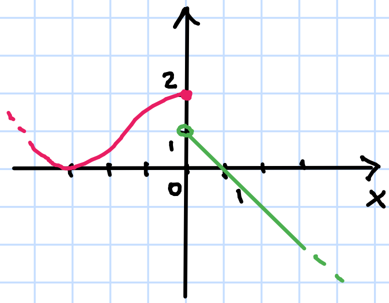
The function f is continuous for all $x \neq 0$ (indeed $\cos(x)+1$ and $-x+1$ are continuous for all $x \in \mathbb{R}$).

So we have only to check the behavior of x at its breaking point $x=0$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos(x)+1 = \cos(0)+1 = 2$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} -x+1 = 1$$

Thus, since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, and they are both finite then we have that $x=0$ is a jump discontinuity.

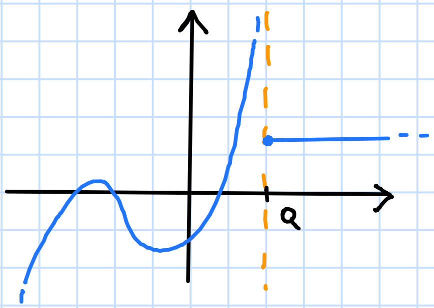


③ INFINITE DISCONTINUITY

Def: A number a is said to be an infinite discontinuity for f if either the left-hand limit or the right-hand limite is infinity, that is:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{and/or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

this means ∞ or $-\infty$ (of course the limit can not be ∞ and $-\infty$ at the same time)

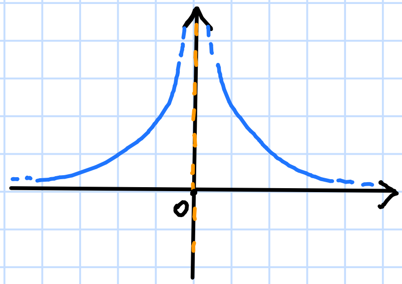


Remark: we do not need that both left-hand and right-hand limits are infinite for a to be an infinite discontinuity ex : $f(x) = \frac{1}{x^2}$

We already saw in class 2 that:

$$\lim_{x \to 0^-} f(x) = \infty = \lim_{x \to 0^+} f(x)$$

Thus $x=0$ is an infinite discontinuity for f.



We have also a notion of continuity from the left and from the right :

Def : A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} f(x) = f(a)$$

A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

Then, a function f is continuous on a closed interval $[c,d]$ if f is continuous from the right at $x=c$ and from the left at $x=d$, that is

$$\lim_{x \to c^+} f(x) = f(c)$$

$$\lim_{x \to d^-} f(x) = f(d)$$

ex : Let us consider one of the previous examples :

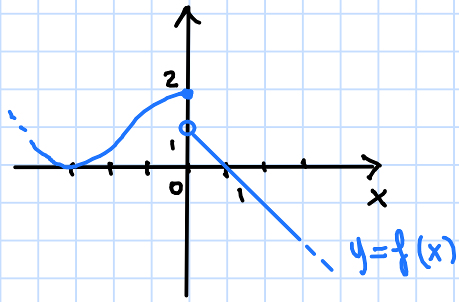
$$f(x) = \begin{cases} \cos(x) + 1, & \text{when } x \leq 0 \\ -x+1, & \text{when } x > 0 \end{cases}$$

We have :

$\lim_{x\to 0^-} f(x) = 2 = f(0) \Rightarrow$ f is continuous from the left at $x=0$

$\lim_{x\to 0^+} f(x) = 1 \neq f(0) \Rightarrow$ f is not continuous from the right at $x=0$.

The graph of f highlights well these two facts.



Remark : When a function f is defined only on one side of an endpoint of an interval, we understand continuous at the endpoint to mean continuous from the left or from the right.

ex: $f(x) = \sqrt{x-1}$

The domain of f is $x-1 \geq 0$, that is $x \geq 1$:

$$D = [1, \infty).$$

f is continuous in D. In particular f is continuous at $x=1$ (from the right) since

$$\lim_{x\to 1^+} f(x) = 0 = f(1).$$

We have the following theorems about continuity.

Theorem : If f and g are continuous at a and c is a constant then the following functions are also continuous at a:

$$f+g, \quad f-g, \quad fg, \quad \frac{f}{g} \text{ if } g(a) \neq 0, \quad cf.$$

sum difference product quotient

Proof: We will prove this statement only for $f+g$. The other proofs will be totally analogous

Proof for $f+g$ (the others are totally analogous)

For showing that $f+g$ is continuous at a, we have to prove that

$$\lim_{x\to a} (f+g)(x) = (f+g)(a).$$

We have:

$$\lim_{x\to a} (f+g)(x) \overset{(f+g)(x) := f(x)+g(x)}{=} \lim_{x\to a} f(x)+g(x) \overset{\text{limit law (sum)}}{=} \lim_{x\to a} f(x) + \lim_{x\to a} g(x) =$$

$$\underset{\substack{f \text{ and } g\\ \text{continuous at}\\ a}}{=} f(a)+g(a) = (f+g)(a)$$

✓ done!

<u>Ex</u>: Let f be a rational function, that is

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

By the previous theorem f is continuous for all x such that $Q(x) \neq 0$.

If a is a number such that $Q(a)=0$, then a is either a removable discontinuity or an infinite discontinuity (depending on the cases).

$$f_1(x) = \frac{x(x-1)}{x-1} \qquad\qquad f_2(x) = \frac{x}{x-1}$$

$x=1$ removable discontinuity (we can "remove" the discontinuity by simplifying $x-1$)

$x=1$ infinite discontinuity (there is no way of simplifying the fraction...)

<u>Theorem 1</u>: If f is continuous at b and $\lim_{x\to a} g(x) = b$ then:

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

the limit moves inside

implies

<u>Theorem 2</u>: If g is continuous at a and f is continuous at $g(a)$ then $f\circ g$ is continuous at a

ex: We want to compute $\lim_{x\to -1} \sin\left(\pi \frac{x+1}{2x^2+2x}\right)$.

If we set:

$f(x) = \sin(x)$

$g(x) = \pi \frac{x+1}{2x^2+2x}$

we have that $\sin\left(\pi \frac{x+1}{2x^2+2x}\right) = f(g(x))$

Now:

$$\lim_{x\to -1} \pi \frac{x+1}{2x^2+2x} = \lim_{x\to -1} \pi \frac{\cancel{x+1}}{2x(\cancel{x+1})} = \frac{\pi}{-2} = -\frac{\pi}{2}$$

and $\sin(x)$ is continuous everywhere (hence in particular at $x = -\frac{\pi}{2}$).

Thus by the previous Theorem 1 we get:

$$\lim_{x\to -1} \sin\left(\pi \frac{x+1}{2x^2+2x}\right) \underset{\text{sin continuous everywhere}}{=} \sin\left(\lim_{x\to -1} \pi \frac{x+1}{2x^2+2x}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$$

Typical exercise on continuity

Find the value(s) of c that make the following function f continuous everywhere:

$$f(x) = \begin{cases} \sin\left(\frac{\pi}{4}x\right) + c, & \text{when } x < 2 \\ x^3 - x - c^2 - 5, & \text{when } x \geq 2 \end{cases}$$

Solution

We remark that:

- $\sin\left(\frac{\pi}{4}x\right)$ is the composition of two continuous functions ($\sin(x)$ and $\frac{\pi}{4}x$) and hence continuous everywhere. Thus $f(x)$ is continuous for all $x < 2$.

- $x^3 - x - c^2 - 5$ is a polynomial and hence continuous everywhere. Thus $f(x)$ is continuous for all $x > 2$.

This implies that f is continuous everywhere if it is also continuous at $x=2$.

Now, f is continuous at 2 if and only if:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

We have:

- $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \sin\left(\frac{\pi}{4}x\right)+c = \sin\left(\frac{\pi}{4}\cdot 2\right)+c = 1+c$
- $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3-x-c^2-5 = 1-c^2$
- $f(2) = 2^3-2-c^2-5 = 1-c^2$

Thus we have to impose that

$$\underbrace{1+c}_{\lim_{x\to 2^-} f(x)} = \underbrace{1-c^2}_{\lim_{x\to 2^+} f(x) = f(2)}$$

I solve the equation

$\iff$ $c^2+c=0 \iff c(c+1)=0 \iff c=0$ or $c=-1$.

We conclude that $c=0$ and $c=-1$ are all the values that make f continuous everywhere.

An important property of continuous functions is expressed by the following theorem:

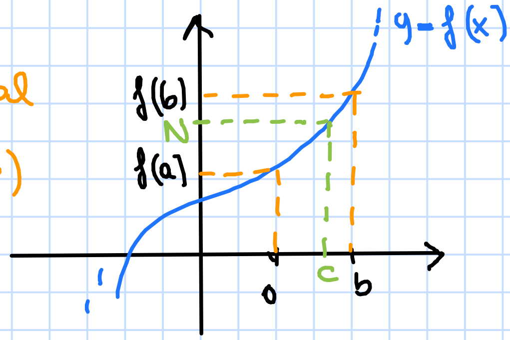
The intermediate value theorem

Let f be a function which is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(b) \neq f(a)$, that is:

$$f(a) < N < f(b) \text{ if } f(a) < f(b)$$
or
$$f(b) < N < f(a) \text{ if } f(b) < f(a).$$

Then there exists a number c in (a,b) such that $f(c)=N$.

For every N in the open interval $(f(a), f(b))$ there is a c in (a,b) such that $f(c) = N$.



Hence the intermediate value theorem guarantees that a continuous function takes on every intermidiate value between the values $f(a)$ and $f(b)$

Ex: • We can understand better the statement of the intermediate value theorem on a concrete situation.

Imagine that a mountaineer is climbing a mountain.

Let $h(t)$ be the function that at each time t (in hours) represents the height of the mountaineer above mean sea level (in feet)

Obviously $h(t)$ is a continuous function (since we can not teleport yet!)

Assume that $h(0) = 500$ feet AMSL and $h(4) = 3000$ feet AMSL.

By the intermediate value theorem, for each height H between 500 feet and 3000 feet AMSL there exists a time t_0 between 0 and 4 hours for which $h(t_0) = H$, (that is, the height of the mountaineer at the time t_0 equals H).

• Also the function of the temperature in a room is continuous.

Hence, if we want to pass from a temperature T_1 to a temperature T_2 by the intermediate value theorem we have to pass trought all the intermediate temperatures between T_1 and T_2.

Typical exercise about the intermediate value theorem

Show that the equation

$$\sin^2\left(\frac{\pi}{4}x\right) = 2-3x^3$$

has at least a solution in $[0,1]$.

Solution

In these kind problems we are not interested in finding the exact value of the solution, but only in showing its existence.

Moreover it is totally possible that the equation admits more than a solution in the interval.

For applying the intermediate value theorem first of all we need a continuous function.

It is clear that the equation

$$\sin^2\left(\frac{\pi}{4}x\right) = 2-3x^3$$

bring all the terms in the same side

is equivalent to the following one:

$$\sin^2\left(\frac{\pi}{4}x\right) + 3x^3 - 2 = 0$$

① Let us set

$$f(x) = \sin^2\left(x\frac{\pi}{4}\right) + 3x^3 - 2$$

this is the function to which we will apply the intermediate value theorem!

Remark that a solution to the previous equation is a number c such that $f(c)=0$.
Hence, all we want to show is that this number c can be found inside $(0,1)$.

② Now $f(x)$ is a continuous function (since sum and composition of continuous functions).

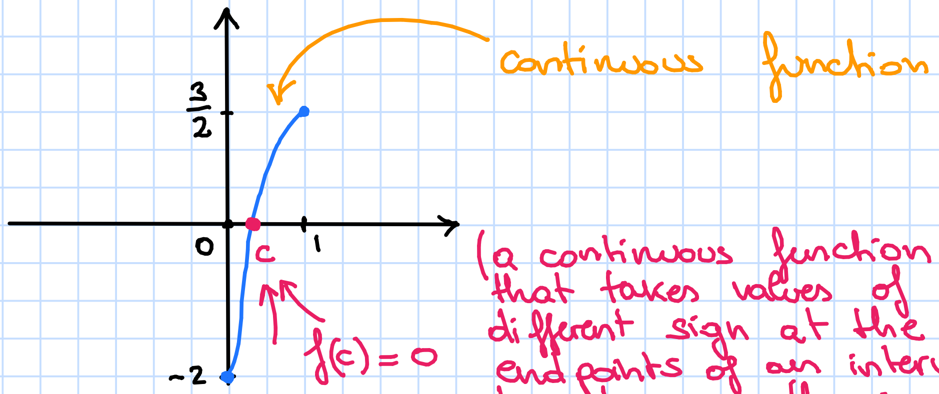
Do not forget to write this! This is the fondamental hypothesis of the intermediate value theorem!

③ Moreover:

$$\begin{cases} f(0) = \sin^2\left(0\cdot\frac{\pi}{4}\right) + 0 - 2 = -2 \\ f(1) = \sin^2\left(1\cdot\frac{\pi}{4}\right) + 3\cdot 1 - 2 = \left(\frac{\sqrt{2}}{2}\right)^2 + 3 - 2 = \frac{1}{2}+1 = \frac{3}{2} \end{cases}$$

compute the value of the function at the endpoints of the interval.

Graphical situation



(a continuous function that takes values of different sign at the end points of an interval has to cross the x-axis somewhere in the interval)

4 Then by the intermediate value theorem for all $-2 < N < \frac{3}{2}$ there exists a number c in $(0,1)$ such that $f(c) = N$.

In particular $-2 < 0 < \frac{3}{2}$, so that there exists c in $(0,1)$ such that $f(c) = 0$, i.e. the equation

$$\sin^2\left(\frac{\pi}{4}x\right) = 2 - 3x^3$$

has at least a solution in $(0,1)$.