Calculus I - MAC 2311 - Section 001

Quiz 6 - Solutions

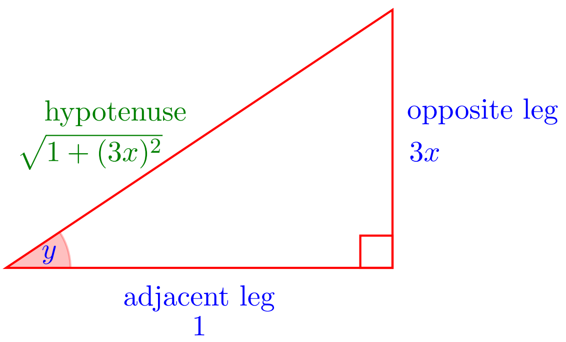
03/21/2018

1) [2 points] Simplify the expression $\sin\left(\tan^{-1}(3x)\right)$.

Solution:

Let us set $y = \tan^{-1}(3x)$. Then $\tan(y) = 3x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

We recall that in a right triangle $\tan(y) = \frac{\text{opposite leg}}{\text{adjacent leg}}$. Here $\tan(y) = \frac{3x}{1}$, hence we can consider the right triangle with opposite leg of length $3x$ and adjacent leg of length 1 (see the picture below):



Hence:

$$\sin\left(\tan^{-1}(3x)\right) = \sin(y) = \frac{\text{opposite leg}}{\text{hypothenuse}} = \frac{3x}{\sqrt{1+9x^2}}.$$

2) [2 points] Compute the derivative of $x^2 \cdot \arcsin(3x^2)$.

Solution:

Let $f(x) = x^2 \cdot \arcsin(3x^2)$. We have:

$$\begin{aligned}
f'(x) &= \left(x^2 \cdot \arcsin(3x^2)\right)' = \\
&= \left(x^2\right)' \cdot \arcsin(3x^2) + x^2 \cdot \left(\arcsin(3x^2)\right)' = \\
&= 2x \cdot \arcsin(3x^2) + x^2 \cdot \frac{1}{\sqrt{1-(3x^2)^2}} \cdot \left(3x^2\right)' = \\
&= 2x \cdot \arcsin(3x^2) + x^2 \cdot \frac{1}{\sqrt{1-9x^4}} \cdot 6x = \\
&= 2x \cdot \arcsin(3x^2) + \frac{6x^3}{\sqrt{1-9x^4}}.
\end{aligned}$$

3) [2 points] Prove that $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$.

Solution:

We set $y = \arcsin(x)$. Then $\sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. We want to prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$. We have:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$
$$\Downarrow$$
$$\cos y \cdot \frac{dy}{dx} = 1$$
$$\Downarrow$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Now, from the Pythagorean identity $\cos^2(y) + \sin^2(y) = 1$, we obtain:

$$\cos^2(y) = 1 - \sin^2(y)$$
$$\Downarrow \cos y \geq 0 \text{ when } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$\cos y = \sqrt{1 - \sin^2 y}$$
$$\Downarrow \sin y = x$$
$$\cos y = \sqrt{1 - x^2},$$

and therefore $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$.

4) [6 points] Compute the following limits:

a) $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin(x)}{\cos(x)}$

Solution:

We have $\lim_{x \to \frac{\pi}{2}} 1 - \sin(x) = 1 - \sin\left(\frac{\pi}{2}\right) = 0$ and $\lim_{x \to \frac{\pi}{2}} \cos(x) = \cos\left(\frac{\pi}{2}\right) = 0$, so we are faced with the indeterminate form $\frac{0}{0}$. Hence we can apply L'Hospital's Rule:

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin(x)}{\cos(x)} = \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin(x))'}{(\cos(x))'} = \lim_{x \to \frac{\pi}{2}} \frac{-\cos(x)}{-\sin(x)} \overset{\text{plug in}}{=} \frac{-\cos\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = \frac{0}{-1} = \mathbf{0}.$$

b) $\lim_{x\to\infty} x^3 e^{-x^2}$

Solution:

We have $\lim_{x\to\infty} x^3 = \infty$ and $\lim_{x\to\infty} e^{-x^2} = 0$, so that we are faced with the indeterminate form $\infty \cdot 0$. Hence we rewrite the limit in the following way:

$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}}$$

Now the indeterminate form is $\frac{\infty}{\infty}$ and we can apply L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{(x^3)'}{\left(e^{x^2}\right)'} = \lim_{x\to\infty} \frac{3x^2}{e^{x^2}\cdot 2x} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{3x}{e^{x^2}\cdot 2}.$$

Again, we are faced with the indeterminate form $\frac{\infty}{\infty}$, therefore we apply a second time L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{3x}{2e^{x^2}} = \lim_{x\to\infty} \frac{(3x)'}{\left(2e^{x^2}\right)'} = \lim_{x\to\infty} \frac{3}{2e^{x^2}\cdot 2x} = \text{“}\frac{1}{\infty}\text{”} = \mathbf{0}.$$

c) $\lim_{x\to\infty} x^{\frac{1}{x}}$

Solution:

We have:

$$\lim_{x\to\infty} x^{\frac{1}{x}} \overset{\text{cancellation equation}}{=} \lim_{x\to\infty} e^{\ln\left(x^{\frac{1}{x}}\right)} \overset{\text{logarithm law}}{=} \lim_{x\to\infty} e^{\frac{1}{x}\ln(x)} \overset{\text{continuity of } e^x}{=} e^{\lim_{x\to\infty}\frac{1}{x}\ln(x)}.$$

Now we compute separately $\lim_{x\to\infty} \frac{1}{x}\ln(x)$:

$$\lim_{x\to\infty} \frac{1}{x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{(\ln(x))'}{(x)'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \text{“}\frac{1}{\infty}\text{”} = 0.$$

Therefore we have:

$$\lim_{x\to\infty} x^{\frac{1}{x}} = e^{\lim_{x\to\infty}\frac{1}{x}\ln(x)} = e^0 = \mathbf{1}.$$